Getting One for Two, or the Contractors' Bad Deal. Towards a Unified Solution to the Semantic Paradoxes[*]

December 23, 2012

1 Three Kinds of Transparent Theories of Truth

Say that a theory of truth is *transparent* iff the theory treats ‘‘φ’ is true’ as *fully intersubstitutable* with φ. Transparency is an appealing *formal principle* about truth, whose force should be equally recognised by very different theories about the *nature* of truth, such as, for example, a broadly *correspondentist* theory and a broadly *deflationist* theory. As for correspondentism, transparency is naturally understood as saying that φ's being true is in a very strong sense both necessary and sufficient for what φ says being the case, and that is in turn something strongly suggested by the claim that φ's being true consists in φ's *corresponding to the facts*. As for deflationism, something like transparency is directly required if the notion

*Earlier versions of the material in this paper have been presented in 2012 at the MCMP Conference on *Paradox and Logical Revision* (Ludwig Maximilian University), at the 5th SPFA Meeting in Braga (University of Minho), at the 10th SIFA Conference in Alghero (University of Sassari), at the LOGOS *Semantic Paradoxes and Vagueness* Seminar (University of Barcelona) and at an NIP workshop in honour of Crispin Wright (University of Aberdeen). I'd like to thank all these audiences for very stimulating comments and discussions. Special thanks go to Massimiliano Carrara, Roy Cook, Philip Ebert, Hartry Field, Branden Fitelson, Luke Fraser, Patrick Greenough, Michiel van Lambalgen, José Martínez, Sebastiano Moruzzi, Julien Murzi, Sergi Oms, Marco Panza, Bryan Pickel, Graham Priest, Stephen Read, David Ripley, Sven Rosenkranz, Gonçalo Santos, Ricardo Santos, Vladimir Stepanov, Alan Weir, Robert Williams, Timothy Williamson, Crispin Wright and an anonymous referee. I'm also very grateful to the editors Theodora Achourioti, Kentaro Fujimoto, Henri Galinon and José Martínez for inviting me to contribute to this volume and for their support and patience throughout the process. In writing the paper, I have benefitted, at different stages, from an AHRC Postdoctoral Research Fellowship and the FP7 Marie Curie Intra-European Fellowship 301493 with project on *A Non-Contractive Theory of Naive Semantic Properties: Logical Developments and Metaphysical Foundations* (NTNSP), as well as from partial funds from the project FFI2008-06153 of the Spanish Ministry of Science and Innovation on *Vagueness and Physics, Metaphysics, and Metametaphysics*, from the project FFI2011-25626 of the Spanish Ministry of Science and Innovation on *Reference, Self-Reference and Empirical Data*, from the project CONSOLIDER-INGENIO 2010 CSD2009-00056 of the Spanish Ministry of Science and Innovation on *Philosophy of Perspectival Thoughts and Facts* (PERSP) and from the FP7 Marie Curie Initial Training Network 238128 on *Perspectival Thoughts and Facts* (PETAF).

of truth is adequately to serve the *expressive needs* that, according to the deflationist, constitute its *raison d'être*. However, plausible as it may seem from a wide variety of perspectives, transparency is not without its problems. I've discussed some of these at great length in Zardini [2008], pp. 545–561; [2012], pp. 260–266; [2013e], arguing that they do place substantial limits on transparency. Here, I'd like to focus on a different and, in some respect, even more fundamental problem for transparency—the *semantic paradoxes*—arguing in favour of a view according to which those paradoxes actually do not place any further limit on transparency.

Consider a standard first-order interpreted language $\mathscr{T}$ that is expressive enough as to contain, for every sentence φ of $\mathscr{T}$, a singular term $\ulcorner\varphi\urcorner$ referring (by some means or other) to φ. Suppose also that T is a predicate of $\mathscr{T}$ expressing the notion of truth. Suppose finally that, in $\mathscr{T}$, t expresses the strongest proposition that logic enjoins to accept (the conjunction of all logical truths) and f the weakest proposition that logic enjoins to reject[1] (the disjunction of all logical falsehoods). The semantic paradoxes historically emerge with the *Liar* paradox in some of its versions (see Bocheński [1970], p. 131), so it just seems fit to begin with that paradox. Let's examine first a particularly canonical version of the paradox, considering a sentence λ identical to $\neg T(\ulcorner\lambda\urcorner)$. We start with:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{reflexivity}
\UnaryInfC{$T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner)$}
\AxiomC{}
\RightLabel{reflexivity}
\UnaryInfC{$T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner)$}
\RightLabel{transparency}
\UnaryInfC{$T(\ulcorner\lambda\urcorner) \vdash \neg T(\ulcorner\lambda\urcorner)$}
\RightLabel{adjunction}
\BinaryInfC{$T(\ulcorner\lambda\urcorner), T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$}
\RightLabel{contraction}
\UnaryInfC{$T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$}
\AxiomC{}
\RightLabel{law of non-contradiction}
\UnaryInfC{$T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner) \vdash \mathsf{f}$}
\RightLabel{transitivity}
\BinaryInfC{$T(\ulcorner\lambda\urcorner) \vdash \mathsf{f}$}
\end{prooftree}
$$

where $\vdash$ expresses the relation of *following from*[2] and, given the prominent feature of the theory I'd like to defend in this paper, premises as well as conclusions are combined into *multisets* (call this reasoning 'A_0'). We continue with:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{reflexivity}
\UnaryInfC{$\neg T(\ulcorner\lambda\urcorner) \vdash \neg T(\ulcorner\lambda\urcorner)$}
\AxiomC{}
\RightLabel{reflexivity}
\UnaryInfC{$\neg T(\ulcorner\lambda\urcorner) \vdash \neg T(\ulcorner\lambda\urcorner)$}
\RightLabel{transparency}
\UnaryInfC{$\neg T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner)$}
\RightLabel{adjunction}
\BinaryInfC{$\neg T(\ulcorner\lambda\urcorner), \neg T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$}
\RightLabel{contraction}
\UnaryInfC{$\neg T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$}
\AxiomC{}
\RightLabel{law of non-contradiction}
\UnaryInfC{$T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner) \vdash \mathsf{f}$}
\RightLabel{transitivity}
\BinaryInfC{$\neg T(\ulcorner\lambda\urcorner) \vdash \mathsf{f}$}
\end{prooftree}
$$

[1] Against an influential tendency in much of the recent literature, I should make clear that by 'reject' and its likes I naturally mean something simply along the lines of *refusal to accept* the relevant proposition (in particular, I mean something that is compatible with thinking that the relevant proposition might be true). I'll use 'deny' and its likes to mean *acceptance of the negation* of the relevant proposition.

[2] Throughout, I use 'follow from' and its relatives to denote the relation of logical consequence (broadly understood so as to encompass also the "logic of truth") while I use 'entail' and its relatives to denote the converse relation. I use 'equivalence' and its relatives to denote two-way entailment. Moreover, I use 'implication' and its relatives to denote the status of the semantic values of two sentences that is necessary and sufficient for the conditional from the one sentence to the other sentence to be true (just like falsehood is the semantic value of a sentence that is necessary and sufficient for the negation of the sentence to be true).

(call this reasoning 'A_1'). We close with:

$$\dfrac{\dfrac{}{\mathsf{t} \ \vdash \ T(\ulcorner\lambda\urcorner) \vee \neg T(\ulcorner\lambda\urcorner)}\text{ law of excluded middle} \qquad \dfrac{\dfrac{}{T(\ulcorner\lambda\urcorner) \ \vdash \ \mathsf{f}}\, A_0 \quad \dfrac{}{\neg T(\ulcorner\lambda\urcorner) \ \vdash \ \mathsf{f}}\, A_1}{T(\ulcorner\lambda\urcorner) \vee \neg T(\ulcorner\lambda\urcorner) \ \vdash \ \mathsf{f}}\text{ reasoning by cases}}{\mathsf{t} \ \vdash \ \mathsf{f}}\text{ transitivity}$$

(call this reasoning 'A_2', and the whole argument 'paradox A'), where I'll assume throughout that $\mathsf{t} \ \vdash \ \mathsf{f}$ is a catastrophe that every theory needs to avoid.

Paradox A is valuable because it makes very clear what are the main moves that a transparent theorist can make, thus serving as a privileged point of entry to a categorisation of the theories I'll discuss in the remainder of this paper: theories that deny the *law of excluded middle* (henceforth, 'LEM'), theories that deny the *law of non-contradiction* (henceforth, 'LNC') and theories that deny the *metarule of contraction* (henceforth, 'contraction'). The focus on the first two is motivated by their intrinsic plausibility and salience in the contemporary debate on the semantic paradoxes; the focus on the third is motivated by the fact that I believe that approach to be the correct one and have elsewhere proposed and developed a specific non-contractive theory (henceforth, for reasons that will become obvious shortly, '**IKT**').[3,4] In the remainder of this paper, I'll undertake a *comparative analysis* of how these three kinds of transparent theories fare with respect to the two most prominent kinds of semantic paradoxes. I'll argue that, at least with respect to those paradoxes, **IKT** has the great advantage of offering a *unified solution*.

Theories denying LEM (henceforth, 'non-LEM theories') have a reasonable intrinsic plausibility when considering a semantic paradox like paradox A, and they have long occupied a leading role in the contemporary debate on the semantic paradoxes (see e.g. Kripke [1975]; Brady [2006]; Field [2008]). Such theories fully accept A_0 and A_1: they thus accept that both

[3] Paradox A also suggests other interesting options that unfortunately there won't be space to discuss in this paper. Let me however remark that merely denying reasoning by cases and accepting all the other principles employed in paradox A does not suffice to uphold transparency. By LEM, $\mathsf{t} \ \vdash \ T(\ulcorner\lambda\urcorner) \vee \neg T(\ulcorner\lambda\urcorner)$ holds, and so, by transparency $\mathsf{t} \ \vdash \ T(\ulcorner\lambda\urcorner) \vee T(\ulcorner\lambda\urcorner)$ holds, and hence, by a version of contraction, $\mathsf{t} \ \vdash \ T(\ulcorner\lambda\urcorner)$ holds. By similar reasoning, $\mathsf{t} \ \vdash \ \neg T(\ulcorner\lambda\urcorner)$ holds, and so, by adjunction, $\mathsf{t} \ \vdash \ T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$ holds. Since, by LNC, $T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner) \ \vdash \ \mathsf{f}$ holds, it follows, by transitivity, that $\mathsf{t} \ \vdash \ \mathsf{f}$ holds. Denial of reasoning by cases is in fact one of the main features of *supervaluationist* and *revision* theories (see e.g. McGee [1991] and Gupta and Belnap [1993] respectively), which are *not* transparent theories. All the theories considered in this paper accept the *specific* version of reasoning by cases employed in paradox A, although I argue in Zardini [2011] that theories denying contraction should also deny a more *general*—and very frequently mentioned—version of that metarule (the issue will briefly crop up in fn 16 and at the end of section 3).

[4] Transparent theories as a whole can in turn be seen as forming a *logical* and *philosophical natural kind* along the correlated dimensions of *how deeply one deviates from classical logic* and, consequently, of *how tightly one can connect truth with reality*. Theories that retain *full classical logic* must give up even the *equivalence* between $T(\ulcorner\varphi\urcorner)$ and φ, and so must allow for the possibility that truth and reality *straightforwardly* come apart. Theories that at least *are closed under classical laws/rules and structural* (but not necessarily operational) *metarules* must still give up the *intersubstitutability* of $T(\ulcorner\varphi\urcorner)$ with φ (i.e. transparency), and so must allow for the possibility that truth and reality come apart *at least in certain contexts* (for example, in the suppositional contexts created by antecedents of conditionals). Transparent theories are characteristic in that, forcing truth and reality to go together *in every context*, they require a *deeper deviation from classical logic* consisting in denying some of its laws/rules or structural metarules. Thanks to Gonçalo Santos for raising this issue.

$T(\ulcorner\lambda\urcorner)$ and its negation are logical falsehoods.[5] Apart from LEM, such theories also accept all the other steps of A_2: in particular, they accept that, if both disjuncts are logical falsehoods, then the disjunction itself is a logical falsehood. Therefore, they deny that the conclusion of the relevant instance of LEM ($T(\ulcorner\lambda\urcorner) \vee \neg T(\ulcorner\lambda\urcorner)$) is a logical truth, and in fact flat-out reject that sentence. They thus deny the unrestricted validity of LEM.[6]

Theories denying LNC (henceforth, 'non-LNC theories') have traditionally been thought to have less intrinsic plausibility even when considering a semantic paradox like the Liar, but they have withstood well the initial incredulous stares and thus have managed to occupy a primary role in the contemporary debate on the semantic paradoxes (see e.g. Priest [2006]; Beall [2009]). Such theories fully accept something even more general than A_2: they accept that if both a sentence and its negation entail a sentence, the latter sentence is a logical truth. Apart from LNC, such theories also accept all the other steps of A_0 and A_1: in particular, they accept that both $T(\ulcorner\lambda\urcorner)$ and its negation entail a contradiction. Therefore, they deny that the premise of the relevant instance of LNC ($T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$) is a logical falsehood, and in fact flat-out accept that sentence. They thus deny the unrestricted validity of LNC.

As for **IKT**, that non-contractive theory technically corresponds to the multiplicative fragment of sentential *affine* logic (i.e. linear logic plus monotonicity) and can very naturally be presented in sequent-calculus style.[7] The *background logic* is defined as the smallest logic containing as axiom the structural rule:

$$\dfrac{}{\varphi \vdash_{\mathbf{IKT}} \varphi}\ \text{I}$$

and closed under the structural metarules:

$$\dfrac{\Gamma_1 \vdash_{\mathbf{IKT}} \Delta}{\Gamma_1, \Gamma_2 \vdash_{\mathbf{IKT}} \Delta}\ \text{K-L} \qquad \dfrac{\Gamma \vdash_{\mathbf{IKT}} \Delta_1}{\Gamma \vdash_{\mathbf{IKT}} \Delta_1, \Delta_2}\ \text{K-R}$$

$$\dfrac{\Gamma_1 \vdash_{\mathbf{IKT}} \Delta_1, \varphi \qquad \Gamma_2, \varphi \vdash_{\mathbf{IKT}} \Delta_2}{\Gamma_1, \Gamma_2 \vdash_{\mathbf{IKT}} \Delta_1, \Delta_2}\ \text{S}$$

[5] Throughout, I assume—plausibly enough in view of our definition of f—that [$\varphi \vdash \mathsf{f}$ holds iff φ itself is a logical falsehood] and—plausibly enough in view of our definition of t—that [$\mathsf{t} \vdash \varphi$ holds iff φ itself is a logical truth]. (Throughout, I use square brackets to disambiguate constituent structure in English.) I offer some justification for the first assumption in fn 19.

[6] As the text already suggests, what I typically have in mind when I talk about "denial of an instance of LEM" is denial because of rejection of its conclusion, rather than denial because the conclusion may fail to have properties over and above truth that are deemed necessary for being a logical truth. Analogous comments apply for my talk below of "denial of an instance of LNC" and "denial of an instance of contraction".

[7] Given that non-LEM and non-LNC theories are already well-known in the literature, a similar presentation of their details would be superfluous. Moreover, given that there are important disagreements of detail among different non-LEM theories and among different non-LNC theories, a single such presentation would be impossible. Finally, given that the arguments of this paper are robust with respect to possible disagreements of detail among different non-LEM theories and among different non-LNC theories, multiple such presentations would be irrelevant.

and under the operational metarules:

$$\frac{\Gamma \ \vdash_{\mathbf{IKT}} \ \Delta, \varphi}{\Gamma, \neg\varphi \ \vdash_{\mathbf{IKT}} \ \Delta}\ \neg\text{-L} \qquad \frac{\Gamma, \varphi \ \vdash_{\mathbf{IKT}} \ \Delta}{\Gamma \ \vdash_{\mathbf{IKT}} \ \Delta, \neg\varphi}\ \neg\text{-R}$$

$$\frac{\Gamma, \varphi, \psi \ \vdash_{\mathbf{IKT}} \ \Delta}{\Gamma, \varphi \,\&\, \psi \ \vdash_{\mathbf{IKT}} \ \Delta}\ \&\text{-L} \qquad \frac{\Gamma_1 \ \vdash_{\mathbf{IKT}} \ \Delta_1, \varphi \qquad \Gamma_2 \ \vdash_{\mathbf{IKT}} \ \Delta_2, \psi}{\Gamma_1, \Gamma_2 \ \vdash_{\mathbf{IKT}} \ \Delta_1, \Delta_2, \varphi \,\&\, \psi}\ \&\text{-R}$$

$$\frac{\Gamma_1, \varphi \ \vdash_{\mathbf{IKT}} \ \Delta_1 \qquad \Gamma_2, \psi \ \vdash_{\mathbf{IKT}} \ \Delta_2}{\Gamma_1, \Gamma_2, \varphi \vee \psi \ \vdash_{\mathbf{IKT}} \ \Delta_1, \Delta_2}\ \vee\text{-L} \qquad \frac{\Gamma \ \vdash_{\mathbf{IKT}} \ \Delta, \varphi, \psi}{\Gamma \ \vdash_{\mathbf{IKT}} \ \Delta, \varphi \vee \psi}\ \vee\text{-R}$$

$$\frac{\Gamma_1 \ \vdash_{\mathbf{IKT}} \ \Delta_1, \varphi \qquad \Gamma_2, \psi \ \vdash_{\mathbf{IKT}} \ \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \ \vdash_{\mathbf{IKT}} \ \Delta_1, \Delta_2}\ \to\text{-L} \qquad \frac{\Gamma, \varphi \ \vdash_{\mathbf{IKT}} \ \Delta, \psi}{\Gamma \ \vdash_{\mathbf{IKT}} \ \Delta, \varphi \to \psi}\ \to\text{-R}$$

IKT can be extended to a *theory of truth* by adding the following metarules for the truth predicate:

$$\frac{\Gamma, \varphi \ \vdash_{\mathbf{IKT}} \ \Delta}{\Gamma, T(\ulcorner\varphi\urcorner) \ \vdash_{\mathbf{IKT}} \ \Delta}\ T\text{-L} \qquad \frac{\Gamma \ \vdash_{\mathbf{IKT}} \ \Delta, \varphi}{\Gamma \ \vdash_{\mathbf{IKT}} \ \Delta, T(\ulcorner\varphi\urcorner)}\ T\text{-R}$$

In Zardini [2011], I investigate in some detail the background-logical and truth-theoretic strength of what is essentially **IKT** (while proving its consistency), especially in the surprisingly many respects of philosophically interesting strength in which it outperforms non-LEM and non-LNC theories (and other theories too; indeed, I do all this for my favoured **IKT**'s extension to the first order). On the background-logical side, it's easy to see that **IKT** validates LEM, LNC, certain specific versions of the metarules of *reductio* (see section 3) and reasoning by cases (see fn 3), the De Morgan equivalences, double-negation introduction and elimination (so that disjunction is definable by conjunction and negation, and conjunction definable by disjunction and negation), the rules of simplification and adjunction for conjunction and of addition and—so I like to call it—"abjunction" (the dual of adjunction) for disjunction,[8] the rule

[8]The metarules given for & and ∨ are essentially the metarules for the "multiplicative" operators tensor and par of linear logics and the "intensional" operators fusion and fission of relevant logics. In both these kinds of logics, such operators are opposed to the "additive" or "extensional" operators, which are typically supposed to express *our informal notions of conjunction and disjunction* (as they occur for example in informal presentations of the semantic paradoxes). Against a likely misunderstanding, I cannot emphasise enough that, with **IKT**'s & and ∨, I intend to give a *theory of precisely our informal notions of conjunction and disjunction* (hence my use of the standard symbolism), and that I take this interpretation to be warranted by the fact the last four rules just mentioned in the text are valid: together, those rules amount to saying that a conjunction is true iff both of its conjuncts are true, and that a disjunction is true iff either of its disjuncts is, which I take to capture the core of our informal notions of conjunction and disjunction. The divergence of my interpretation with the interpretation typically given for linear and relevant logics is explainable by the fact that those logics lack K-L and K-R, which determines that at least some of the rules just mentioned in the text are not valid for their "multiplicative" or "intensional" operators (hence these logics' search for other operators that can represent more adequately our informal notions of conjunction and disjunction).

of *modus ponens* and the deduction theorem for the conditional (plus other principles that are characteristic of the material conditional, like the so-called 'paradoxes of material implication', so that the conditional is in effect definable by conjunction and negation or by disjunction and negation). Moreover, it's also easy to see that contraction can be locally recovered for a set of sentences X by adding, for every $\varphi \in X$, $\varphi \to \varphi \,\&\, \varphi$[9] as a further axiom to the theory. On the truth-theoretic side, it's easy to see that **IKT** validates the equivalence between $T(\ulcorner\varphi\urcorner)$ and φ, the Tarskian biconditionals, transparency, the standard truth-functional laws and the traditional constraint of truth preservation on logical consequence. I refer the reader to Zardini [2011] for a sustained philosophical and technical study of **IKT** (see also Zardini [2013a]; [2013b]; [2013c]; [2013d] for studies of various issues of detail), and in this paper focus instead on another important aspect in which I think **IKT** enjoys a great advantage over non-LEM and non-LNC theories—that is, in promising a unified solution to the semantic paradoxes.

IKT fully accepts A_2: it thus accepts that if both a sentence and its negation were logical falsehoods, a catastrophe would ensue. Apart from contraction, **IKT** also accepts all the other steps of A_0 and A_1: in particular, it accepts that both $T(\ulcorner\lambda\urcorner), T(\ulcorner\lambda\urcorner)$ and $\neg T(\ulcorner\lambda\urcorner), \neg T(\ulcorner\lambda\urcorner)$ entail a contradiction and so that they entail f. Therefore, it denies that $T(\ulcorner\lambda\urcorner)$ entailing f ($\neg T(\ulcorner\lambda\urcorner)$ entailing f) can be validly inferred from $T(\ulcorner\lambda\urcorner), T(\ulcorner\lambda\urcorner)$ entailing f (from $\neg T(\ulcorner\lambda\urcorner), \neg T(\ulcorner\lambda\urcorner)$ entailing f), and in fact flat-out denies the former entailments and flat-out accepts the latter entailments. It thus denies the unrestricted validity of contraction.

2 Some Historical Background on Non-Contractive Approaches to Paradox

Before examining how the three kinds of theories presented in section 1 deal with another version of the Liar paradox and with the other most prominent kind of semantic paradox, I'd like to provide some brief historical background on the most significant historical antecedents of the non-contractive solution to the semantic paradoxes that I'm defending. I actually don't know of anywhere in the literature where restriction of contraction is the key component in a *philosophically motivated approach specifically focussed on the semantic paradoxes*. There are though a couple of *logical* and *computer-science* (rather than *philosophical*) traditions that have worked on the technical details of certain non-contractive logics, and have typically applied these to the *set-theoretic* (rather than *semantic*) paradoxes.[10]

The most relevant tradition is probably that represented by BCK *set theories* (whose study has been initiated by Grišin [1974]). Very regrettably, it would appear that that paper is only in Russian with no translation available into English, with even the Russian version not being easily accessible. Because of these circumstances, a more informed study of the technical points of similarity and dissimilarity between the two theories will have to wait. However, so much

[9] To save on brackets, I will assume the usual scope hierarchy among the operators (with $\neg$ binding more strongly than & and $\vee$, and with these in turn binding more strongly than $\to$) and right associativity for each 2ary operator (so that $\varphi_0 \star \varphi_1 \star \varphi_2 \ldots \star \varphi_i$ reads $\varphi_0 \star (\varphi_1 \star (\varphi_2 \ldots \star \varphi_i)))\ldots)$, with $\star$ being a 2ary operator).

[10] A fascinating earlier reference envisaging failure of contraction (but not in relation to the paradoxes) may be Pastore [1936], a work that however requires further historical and exegetical investigation. Thanks to Luke Fraser for alerting me to the existence of Pastore's work.

can already be said. Grišin's theory includes both *multiplicative* connectives and *additive* ones (see fn 8), without specifying which (if any) are supposed to express our informal notions of conjunction and disjunction (this is probably due to the more general lack of philosophical underpinnings of the paper; it is precisely the focus on the multiplicative operators that gives **IKT** much of its logical strength—for example, the validity of LEM and LNC—and so much of its philosophical interest). Relatedly, touching on an issue that I'll have to leave in the background in this paper, Grišin's theory only includes additive *quantifiers* (as defined for example by the standard metarules adopted in the sequent calculus for classical logic and many other logics), which, as I've argued in Zardini [2011], pp. 510–511, yield an objectionable theory of quantification.[11]

As I said, both antecedent non-contractive traditions tend to be concerned with sets rather than truth, and Grišin's theory is no exception (see however Stepanov [2007] for a purely technical, non-philosophical non-contractive approach, which unfortunately includes no theory of conjunction, disjunction or quantification). That tendency is something this paper aims to correct: a non-contractive logic fits transparent truth in a surprisingly *optimal* fashion, getting us what we want and getting us rid of what we don't want (as I'll argue for important aspects in this paper and as I've argued for other aspects in my work referenced in section 1), and promises a unified solution to its paradoxes (as I'll argue in this paper). Such features do not seem to be replicated in the case of the application of a non-contractive logic to the naive theory of sets, where restriction of contraction does not seem to yield a satisfactory theory. Grišin's own theory serves as an exemplification of the problem, since, in merely restricting contraction, while it manages to uphold the principle of full *comprehension*, it does not seem to manage to retain consistency with the principle of full *extensionality* (as is realised in Grišin [1981]), which is arguably essential to sets.[12]

Similar points apply to the couple of theorists that have followed Grišin on all these questionable choices (White [1987]; [1993]; Petersen [2000]; [2002], all of which actually develop theories that are even weaker than Grišin's in some philosophically crucial respect). That said, I should like to add that, contrary to all other authors mentioned in this section, Petersen's theory has rich philosophical foundations that I hope in future work to be able to discuss and compare with the picture I sketch at the end of section 4.

The second, more loosely connected tradition I'd like to mention is that represented by *linear logics* (whose study has been initiated by Girard [1987]). These logics have all the drawbacks I've mentioned about the background logic of BCK set theories. Additionally, they are substantially weaker in that they do not validate the metarule of *monotonicity*, which, as I've mentioned in fn 8, plays a crucial role in securing the strength of my theory. Much of the work done in linear logics is actually focussed on certain 1ary modal connectives, called '*exponentials*', which, while unexceptionable (and indeed very interesting and useful) from a purely logical and technical point of view, have no space in my theory, as I hope to be able to

[11] Although I cannot go into the details in this paper, the latter difference leads to Grišin's theory remaining *finitary*, while mine goes *infinitary*, since it introduces quantifiers by means of metarules akin to the ω-rule; the latter difference also leads to major divergences in the *deductive systems* codifying the respective theories and in the consistency proofs based on those (see Zardini [2011], pp. 511–512, 524–532).

[12] Ultimately, I don't think that Grišin's result forecloses a non-contractive naive theory of sets, but the issue lies beyond the scope of this paper.

discuss in future work. Some authors have studied whether and to what extent linear logics can be used as background logics for set theories based on the principle of full comprehension (see e.g. Girard [1998]). However, exactly because of the presence of exponentials, while shedding light on important *computational issues*, the results have been rather disappointing for those in seek of a viable *theory of sets*.

3 A Unified Solution to the Liar Paradox

The Liar paradox can be introduced in an interestingly different—and equally common—way. We start with:

$$\dfrac{\dfrac{}{T(\ulcorner\lambda\urcorner)\ \vdash\ \mathsf{f}}\,\mathrm{A}_0}{\mathsf{t}\ \vdash\ \neg T(\ulcorner\lambda\urcorner)}\ \text{single-premise reduction theorem}$$

(call this reasoning ‘B_0’). We continue with:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{t}\ \vdash\ \neg T(\ulcorner\lambda\urcorner)}\,\mathrm{B}_0}{\mathsf{t}\ \vdash\ T(\ulcorner\lambda\urcorner)}\ \text{transparency}\qquad \dfrac{}{\mathsf{t}\ \vdash\ \neg T(\ulcorner\lambda\urcorner)}\,\mathrm{B}_0}{\mathsf{t},\mathsf{t}\ \vdash\ T(\ulcorner\lambda\urcorner)\ \&\ \neg T(\ulcorner\lambda\urcorner)}\ \text{adjunction}}{\mathsf{t}\ \vdash\ T(\ulcorner\lambda\urcorner)\ \&\ \neg T(\ulcorner\lambda\urcorner)}\ \text{contraction}$$

(call this reasoning ‘B_1’).[13] We close with:

$$\dfrac{\dfrac{}{\mathsf{t}\ \vdash\ T(\ulcorner\lambda\urcorner)\ \&\ \neg T(\ulcorner\lambda\urcorner)}\,\mathrm{B}_1\qquad \dfrac{}{T(\ulcorner\lambda\urcorner)\ \&\ \neg T(\ulcorner\lambda\urcorner)\ \vdash\ \mathsf{f}}\,\mathrm{LNC}}{\mathsf{t}\ \vdash\ \mathsf{f}}\ \text{transitivity}$$

(call this reasoning ‘B_2’, and the whole argument ‘paradox B’).[14]

A major feature of paradox B is that it does not employ LEM. This is not to say that paradox B has typically been taken to tell against non-LEM theories, for these theories, while accepting all the other principles employed in paradox B, would typically deny the metarule of the *single-premise reduction theorem* (if $\varphi \vdash \mathsf{f}$ holds, $\mathsf{t} \vdash \neg\varphi$ holds). A discussion of the single-premise reduction theorem (and of its relationship to *reductio*) is thus in order. Consider first *reductio*. This is well-known to come at least in two versions: a *distinctively classical* version (if $\Gamma, \neg\varphi \vdash \Delta, \varphi$ holds, $\Gamma \vdash \Delta, \varphi$ holds) and an *intuitionistically acceptable*

[13] Contraction on t is admissible even in **IKT** and so I won’t henceforth bother to make it explicit.

[14] While paradox A corresponds to the kind of informal presentation of the Liar paradox that proves that the Liar sentence cannot be true and proves that the Liar sentence cannot be untrue (while observing that it must be either true or untrue), paradox B corresponds to the kind of informal presentation of the Liar paradox that first proves that the Liar sentence is untrue and then on that basis proves that it is also true (while observing that it cannot be both true and untrue).

version (if $\Gamma, \varphi \vdash \Delta, \neg\varphi$ holds, $\Gamma \vdash \Delta, \neg\varphi$ holds).[15] Now, any transparent theory generates so much *impredicativity* as to yield *ungrounded sentences that are strongly equivalent with their own negation*—sentences that in some sense consist in their self-negation (λ being a standard example of this). It is for this reason that, in the theoretic context of transparent theories, classical and intuitionist *reductio* immediately look as rather silly principles: in that context, it becomes one of the most distorting features of classical and intuitionist logic that *they rule out the very existence of such sentences*, and, at least barring the possibility of true contradictions, classical and intuitionist *reductio* are already sufficient to yield that most distorting result. More generally, in the theoretic context of transparent theories, even allowing for the possibility of true contradictions classical and intuitionist *reductio* still yield that such sentences can only exist if some contradictions are true—a result which is, without further justification, very dubious at best. Indeed, in that context, even allowing for the possibility of true contradictions the inference from the *intensional* fact that an ungrounded sentence is strongly equivalent with its own negation to the *categorical* fact that that sentence (and its negation, and the negation of its negation etc.) holds comes across, without further justification, as an egregious *non sequitur*: in that context, it seems perfectly coherent to accept the intensional fact—which simply determines what the ungrounded content of the sentence is—while rejecting the categorical facts—which would determine whether such content holds.[16,17]

Let's now move on to consider the reduction theorem. Notice first that non-LEM and non-LNC theories unfortunately turn out to be banned from accepting close kins of the single-premise reduction theorem (and turn out to be so banned independently of the semantic paradoxes). As for non-LEM theories, by reflexivity, $\varphi \vdash \varphi$ holds, and so, by a particularly uncontroversial application of monotonicity, $\varphi \vdash \varphi, \mathtt{f}$ holds, and hence, by the *multiple-conclusion reduction theorem* (if $\varphi \vdash \Delta, \mathtt{f}$ holds, $\mathtt{t} \vdash \Delta, \neg\varphi$ holds), $\mathtt{t} \vdash \varphi, \neg\varphi$ holds. By disjunction in the conclusions, LEM follows. Since non-LEM theories typically accept all the

[15] Having officially recorded the versions with side-premises and side-conclusions, for simplicity I'll ignore these in my treatment of *reductio*.

[16] Even adding the further disjunctive assumption that either the ungrounded sentence or its negation holds (an instance of LEM) does not substantially improve the appeal of the inference in the theoretic context of transparent theories. For, in that context, to suppose that the sentence holds is to suppose something intuitively equivalent with its failing to hold (since the sentence actually denies of itself that it holds), and so the fact that the suppositions represented by the two disjuncts can both be so developed as to reach the supposition that the sentence holds is after all not great evidence in favour of the sentence holding (since the supposition so reached is an unstable one). Thanks to Sven Rosenkranz for pressing me on this issue.

[17] Classical *reductio* is traditionally known as '*consequentia mirabilis*' (or 'Clavius' law', from the (Latin) name of the Counter-Reformation Jesuit priest who, actually in the steps of Girolamo Cardano, did much for promoting this method of proof in mathematics). The two names betray two completely different understandings of the principle. The *modern* name signals an understanding of the principle under which it *implicitly involves deriving the contradiction* $\varphi \,\&\, \neg\varphi$ from $\neg\varphi$, and from this fact inferring φ (presumably with the implicit thought that the first derivation shows that $\neg\varphi$ is false, which, in a bivalent spirit, is then taken to suffice for φ's being true). The traditional name signals an understanding of the principle under which it only involves deriving φ from $\neg\varphi$, and from this fact inferring φ (presumably with the implicit thought that the first derivation shows that, even if $\neg\varphi$ is true, φ still is, which, in a bivalent spirit, is then taken to suffice for φ's being true), *without any derivation of a contradiction justifying the reasoning* (one can see the two understandings nicely opposed in an epistolary debate between Christiaan Huygens and André Tacquet; see Nuchelmans [1992] for a historical reconstruction of the dispute). Neither understanding helps in mitigating the objections against the principle levelled in the text. I discuss how both understandings covertly assume contraction at the end of this section.

other principles employed in this reasoning, they must deny the multiple-conclusion reduction theorem. A dual point can be made for non-LNC theories. By reflexivity, $\varphi \vdash \varphi$ holds, and so, by a particularly uncontroversial application of monotonicity, $\varphi, \mathsf{t} \vdash \varphi$ holds, and hence, by the *multiple-premise demonstration theorem* (if $\Gamma, \mathsf{t} \vdash \varphi$ holds, $\Gamma, \neg\varphi \vdash \mathsf{f}$ holds), $\varphi, \neg\varphi \vdash \mathsf{f}$ holds. By conjunction in the premises, LNC follows. Since non-LNC theories typically accept all the other principles employed in this reasoning, they must deny the multiple-premise demonstration theorem.

Let's grant for the sake of argument that non-LEM and non-LNC theories can make good sense of their denial of close kins of the single-premise reduction theorem; after all, as I've already mentioned, the problems raised in the previous paragraph do not rely on any truth-theoretic principle, and so they are much more general problems affecting virtually all theories that deny LEM or LNC (such as intuitionist logic and dual intuitionist logic respectively). The problem becomes much more specific with the single-premise reduction theorem. Contrary to classical and intuitionist *reductio* and to the multiple-conclusion reduction theorem, the input of the single-premise reduction theorem is not that a sentence entails its own contradictory or that it entails f as a side-conclusion—it is the intuitively much stronger one that a sentence entails f, and so that it is a logical falsehood.[18] Even on a transparent theory, the single-premise reduction theorem would seem irresistible: if a sentence is a logical falsehood, certainly it is false as a matter of logic and so its negation is true as a matter of logic and hence a logical truth—in other words, just like we have the principle of *truth functionality* that [the negation of a sentence is true if the sentence is false], we also have the principle of *logical-truth functionality* that [the negation of a sentence is logically true if the sentence is logically false]. The bad news for non-LEM theories is that, compelling as these considerations may seem, they ought to be rejected by a non-LEM theory on pain of paradox B.

There is an apparently terminological issue with the way the argument in the previous paragraph in favour of the single-premise reduction theorem has been presented which may be worth addressing. The argument plausibly assumes (as per fn 5) that *sentences entailing* f are *logical falsehoods*. The apparently terminological issue concerns the likely protest of a non-LEM theorist to the effect that the epithet 'logical falsehood' should be reserved for *sentences whose negation is a logical truth*. In fact, while the two properties are co-extensional in classical logic and in **IKT**, they come apart in both non-LEM theories (in particular, in these theories, a sentence can entail f while its negation is not a logical truth) and non-LNC theories (in particular, in these theories, the negation of a sentence can be a logical truth while the sentence does not entail f). Let's grant (only for the rest of this paragraph) this apparently terminological point about 'logical falsehood' and let's label instead with 'inconsistent' sentences entailing f.[19] That does not improve much the situation for non-LEM

[18] Because of this, in Zardini [2011], p. 514, fn 38 I've argued also for the terminological point that the traditional name '*reductio ad absurdum*' for what I'm in this paper calling '*reductio*' is an egregious misnomer. For a metarule properly called '*reductio ad absurdum*' should have an input saying that certain premises lead to an absurdity, which is clearly the case for the single-premise reduction theorem and clearly not the case for either classical or intuitionist *reductio*, which would more properly be called '*reductio ad ipsius contradictoriam*'.

[19] As might have been intimated by the qualification 'apparently', I don't think that the issue is purely terminological. I think we can quite convincingly argue in a variety of ways that logical falsehood just is inconsistency. A first argument for that conclusion mimics the standard argument against the definition of falsehood of a sentence in terms of truth of its negation, which consists in the simple observation that that

theories, for the argument in the previous paragraph in favour of the single-premise reduction theorem can be recast by saying that, if a sentence is inconsistent, certainly it cannot$_L$[20] be true and so must$_L$ be untrue and hence, by the full power of transparency, its negation must$_L$ be true and thus is a logical truth—the force of the argument remains strong and independent of our getting to attach the label 'logical falsehood' to inconsistent sentences. I suppose that the non-LEM theorist will have to deny the inference from 'φ is inconsistent' to 'φ cannot$_L$ be

definition gets things dramatically wrong *for sentences belonging to languages which don't have negation.* In a completely analogous fashion, we can argue against the definition of logical falsehood of a sentence in terms of logical truth of its negation, by simply observing that that definition gets things dramatically wrong *again for sentences belonging to languages which don't have negation.* And, if the logical falsehood of a sentence is not to be identified with the logical truth of its negation, it's hard to see *what else* is left for it to be other than the inconsistency of the sentence. A second argument for the conclusion starts from the observation that inconsistent sentences behave dually with respect to logical truths, in the sense that, just like logical truths correspond to valid *no-premise, single-conclusion* arguments, inconsistent sentences correspond to valid *single-premise, no-conclusion* arguments. Assuming very plausibly that *logical falsehood behaves dually with respect to logical truth*, that forces the identification of logical falsehood with inconsistency. Or, similarly, just as the *best judgement that logic can pass on a sentence in itself* (i.e. not *qua* component of more complex sentences) is that it is the conclusion of a valid no-premise, single-conclusion argument, so the *worst judgement that logic can pass on a sentence in itself* is that it is the premise of a valid single-premise, no-conclusion argument. Assuming very plausibly that, just as the best judgement that logic can pass on a sentence in itself is equal to a judgement of logical truth, so *the worst judgement that logic can pass on a sentence in itself is equal to a judgement of logical falsehood*, that again forces the identification of logical falsehood with inconsistency. A third argument for the conclusion (and indeed, as we'll see, for something even stronger) assumes very plausibly that just as *(logical) truth is closed under entailment* (at least in the sense that, if φ is a (logical) truth and φ entails ψ, it follows that ψ is a (logical) truth), so *(logical) falsehood is closed under logical consequence* (at least in the sense that, if ψ is a (logical) falsehood and ψ follows from φ, it follows that φ is a (logical) falsehood). The argument then bifurcates. As for non-LEM theories, we only need to establish that inconsistency implies logical falsehood. We do so by observing that, at least in the systems of interest for this paper, being inconsistent implies *entailing sentences that are (logical) falsehoods by anyone's lights* (for example, 'For every P, P'), from which the desired implication follows by closure of (logical) falsehood under logical consequence (and, as I'll observe in the text, the further result just implicitly established that inconsistent sentences are merely false is already incompatible with non-LEM theories). As for non-LNC theories, we only need to establish that logical falsehood implies inconsistency. We do so by reducing to absurdity the claim that (it doesn't because) some logical falsehoods are also logical truths. In turn, we do so by observing that, at least in the systems of interest for this paper, being a logical truth implies *being entailed by sentences that are not (logical) falsehoods by anyone's lights* (for example, 'For some P, P'), from which the desired absurdity follows by closure of (logical) falsehood under logical consequence (and, as I'll observe in the fn 33, the further result just implicitly established that logical truths lack mere falsehood is already incompatible with non-LNC theories). (As a matter of fact, non-LEM and non-LNC theorists are wont to reject closure of falsehood under logical consequence. The high implausibility of rejecting such a fundamental principle connecting logical consequence with falsehood is typically masked by recommending an apparently similar principle of closure of *rejectability* under logical consequence (if ψ ought to be rejected and ψ follows from φ, it follows that φ ought to be rejected). The recommendation is both disappointing and wrong. The recommendation is disappointing as it offers a *superficial, merely normative Ersatz* talking about *what people ought to do* in substitution for a *deep, fully descriptive* principle connecting logical consequence with *semantics* and so with *the ways the objective world can be* independently of people and of what they ought to do (in fact, even if it were true, the normative principle would cry out for a deeper explanation appealing, among other things, to something along the lines of the descriptive principle). The recommendation is also wrong as virtually every *counterexample to closure of knowledge and justification* can be turned into a counterexample to closure of acceptability and rejectability—those normative notions, as opposed to the semantic notions of truth and falsehood, are not suitable for the formulations of appropriate closure principles.)

[20]Throughout, modals and their likes subscripted with 'L' express logical modality.

true' just as she had to deny the inference from 'φ is inconsistent' to 'φ is a logical falsehood'.[21] The non-LEM theorist has thus to deny that *even the worst judgement* that logic can pass on a sentence in itself suffices for the necessary$_L$ untruth of that sentence—indeed, it's easy to see that the non-LEM theorist even has to deny that it suffices for the mere untruth of that sentence (in this sense, although logic would still be powerful enough to establish many truths, somehow it would no longer be powerful enough to establish any untruths). Such a *lack of connection* between *logical consequence* on the one hand and *negation* and *logical modality* (and even *mere* negation) on the other hand strikes me as a great cost of these theories.[22]

Obviously, the foregoing dialectic concerning the single-premise reduction theorem could profitably be continued. Let's stop here however since, for the purposes of this paper, the point is not so much to argue against the non-LEM theorist's denial of the single-premise reduction theorem, but only to establish the weaker point that the non-LEM theorist's solution to paradox B consisting in the denial of the single-premise reduction theorem *does not flow* from her solution to paradox A consisting in the denial of LEM. The latter point can easily be established by noticing that denial of LEM is quite compatible with acceptance of the single-premise reduction theorem: *intuitionist* logic provides a prime example of a coherent logical system lacking the former but having the latter. Thus, whatever rationale the non-LEM theorist might eventually come up with in order to justify her denial of the single-premise reduction theorem, it cannot merely consist in her denial of LEM—in fact, our discussion of the single-premise reduction theorem already amply shows that such a rationale would have to appeal to some rather unobvious considerations concerning the *lack of connection* between *logical consequence* on the one hand and *negation* and the accompanying notions of *(necessary$_L$) untruth* and *(logical) falsehood* on the other hand that are quite foreign to the issue as to whether LEM is valid.

At this point, the non-LEM theorist might grant that denial of LEM does not offer a unified solution to the semantic paradoxes, but suggest that her unified solution comes rather from whatever turns out to be the fundamental thought behind her solution to paradox B

[21] Might she not deny instead the *duality* of necessity$_L$ and possibility$_L$? In our dialectical context, the move would be extremely problematic in several respects. Firstly, short of transparency failing in contexts of logical modality, the move in effect now accepts the metarule from $\varphi \vdash \mathsf{f}$ to $\mathsf{t} \vdash$ 'It is not possible$_L$ that φ'. If in addition we still have the extremely plausible rule φ, 'It is not possible$_L$ that φ' $\vdash \mathsf{f}$, the resulting non-LEM theory is trivial (see the final version of paradox B mentioned in section 4). Secondly, the duality of necessity$_L$ and possibility$_L$ is entailed by the duality of universal quantification and particular quantification plus the standard modality-worlds-linking principles 'It is necessary$_L$ that φ iff, for every possible$_L$ world w, 'φ' is true at w' and 'It is possible$_L$ that φ iff, for some possible$_L$ world w, 'φ' is true at w' (together with an appropriate version of transparency for 'true at w' like 'In w, 'φ' is true at w iff φ' and with the auxiliary assumption ''φ' is true at w_0 iff, for every possible$_L$ world w_1, in w_1, 'φ' is true at w_0'). Thirdly, the contrapositive of 'If φ, then it is possible$_L$ that φ' suffices to license the inference from ''φ' cannot$_L$ be true' to ''φ' is not true', which, as I'll observe in the text, is already incompatible with non-LEM theories. Thanks to Robert Williams for urging me to consider these issues.

[22] I should note that the argument in the text does not fall afoul of the worry I raised above regarding the inference from intensional facts to categorical ones. For that worry concerned the specific case in which the intensional fact is that an ungrounded sentence is strongly equivalent with its own negation, while the different specific case in which the intensional fact is that a sentence entails f is a case in which the intensional fact itself already intuitively involves, if not even is constituted by, the categorical fact that the sentence is a logical falsehood (or by the categorical fact that the sentence is inconsistent). Thanks to Sven Rosenkranz and an anonymous referee for raising this issue.

consisting in the denial of the single-premise reduction theorem. Let's assume, plausibly, that such fundamental thought is the idea that *some sentences are such that both they and their negation are logical falsehoods*. Let's set aside that, when so bluntly presented, the new view enjoys considerably less intrinsic plausibility than the original view that LEM is not valid. If that fundamental thought is correct, it provides a rationale for denying the single-premise reduction theorem (and thus solving paradox B): if both φ and $\neg\varphi$ are logical falsehoods, the single-premise reduction theorem implies the multiply repugnant conclusion that $\neg\varphi$ and $\neg\neg\varphi$ are logical truths. The fundamental thought also provides a rationale for denying LEM (and thus solving paradox A): if both φ and $\neg\varphi$ are logical falsehoods, reasoning by cases $\varphi \vee \neg\varphi$ is also a logical falsehood, and so presumably not a logical truth.

Have we thus hit on a more appropriate formulation of non-LEM theories, one that allows for a unified solution to the semantic paradoxes? Emphatically no. Firstly, and less conclusively, the new fundamental thought is *not at all distinctive* of non-LEM theories: for example, *supervaluationist* and *revision* theories that do accept LEM and deny transparency (see fn 3) would agree that some sentences are such that both they and their negation are logical falsehoods.[23] On the conception in question, the *distinctiveness* of non-LEM theories against all the other main alternative theories would not consist in what would be the *fundamental thought* affording them a unified solution to the semantic paradoxes (i.e. the thought that some sentences are such that both they and their negation are logical falsehoods); it would rather consist in (that thought plus) an *ancillary thought* about disjunction (i.e. the thought that LEM fails for all sentences such that both they and their negation are logical falsehoods). The two thoughts need not of course be completely unrelated—I've mentioned in the previous paragraph how the fundamental thought can be used to yield the ancillary one given plausible additional assumptions about disjunction.[24] But the sore point remains that, on the conception in question, the fundamental thought affording to a non-LEM theory a unified solution to the semantic paradoxes is divorced from the ancillary thought characterising it against the all the other main alternative theories. Notice that, in general, the fact that a theory shares its fundamental thought with a theory differing from it in some other respects is of course just to be expected. What was not expected was that non-LEM theories and, say, supervaluationist theories—one of their traditional arch-rivals—would stand in such a relationship of being merely different variations on exactly the same theme, and so that non-LEM theories' take on the essence of the semantic paradoxes would be something wholeheartedly endorsed by some of the other main alternative theories.[25] Moreover, as we'll appreciate over

[23]Independently of the issue of which formulation of non-LEM theories is most appropriate, the observation in the text shows that a non-LEM theory becomes *completely indistinguishable* from supervaluationist and revision theories in a language lacking disjunction (and the resources to define it) but expressive enough as to generate semantic paradoxes (the language needed to generate the essence of paradox B is a good example of such a language).

[24]Such assumptions start to look less plausible *vis-à-vis* the alternative assumptions made by supervaluationist and revision theories once it is realised that, keeping fixed the duality of conjunction and disjunction and the idea that some sentences are such that both they and their negation are logical falsehoods, the latter assumptions allow supervaluationist and revision theories to uphold the very plausible claim that the negation of any contradiction is a logical truth, while the former assumptions force non-LEM theories to endorse the very implausible claim that the negations of certain contradictions are logical falsehoods.

[25]Contrast with **IKT**, whose fundamental thought that some sentences are such that they fail to contract is strong enough to characterise it against all the other main alternative theories. Of course, **IKT** too has

the next few paragraphs, this problem has the tendency to degenerate badly—especially with respect to the connection between the fundamental thought and the ancillary thought—once it interacts with another somewhat converse problem to which we now turn.[26]

Secondly, and more conclusively, not every semantic paradox has truck with even apparent logical falsehoods: the *truth-teller* paradox ('This sentence is true', see Mortensen and Priest [1981]), the *no-no* paradox ('The next sentence is not true', 'The previous sentence is not true', see Sorensen [2001], pp. 165–170), certain versions of *Epimenides'* paradox ('This sentence is not true and the number of stars in the universe is not even', see Goldstein [1986]) are paradigmatic examples of *semantic paradoxes without even apparent logical falsehoods*.[27] The natural solution given to these paradoxes by non-LEM theories consists in rejecting the relevant excluded middles[28] (and hence denying LEM), rather than in claiming that the relevant sentences are such that both they and their negation are logical falsehoods.[29]

In response, one could, I suppose, stretch the notion of "logic" at play in a non-LEM theory so that every sentence rejected by the theory counts as a "logical falsehood". But such a proposal would be unsatisfactory for various related reasons. Firstly, in the new stretched sense of 'logical falsehood', the apparently *objective* claim that some sentences are such that both they and their negation are "logical falsehoods" boils down to the *autobiographical* claim that some sentences are such that both they and their negation are rejected by the theory (the theorist really), which can hardly be taken to be a fundamental thought behind a solution to the semantic paradoxes. In fact, such a claim immediately cries out for a deeper explanation: why does the theory reject both a sentence and its negation? Secondly, the proposal does nothing but *exacerbate* the problem discussed two paragraphs back of distinguishing the fundamental thought of non-LEM theories from what the other main alternative theories endorse: *virtually all theories*—including my favoured interpretation of **IKT**—that accept the equivalence between $T(\ulcorner\varphi\urcorner)$ and φ but do not accept both a sentence and its negation agree in rejecting the relevant paradoxical sentences and their negation. Thirdly, the ancillary thought about disjunction concerning the failure of LEM *does no longer even remotely follow from the fundamental thought as so weakly understood*: usually, rejecting both disjuncts is no sufficient ground for rejecting a disjunction (as evidenced by ordinary cases of *agnosticism*).

its own variations. In Zardini [2013a], I study in some detail a particularly natural one that replaces the *multiplicative* operators with the *additive* ones, and argue that such variation suffers from a *lack of connection* between *logical consequence* on the one hand and *conjunction* and *disjunction* on the other hand similar to the *lack of connection* between *logical consequence* on the one hand and *negation* on the other hand suffered by non-LEM theories.

[26]Thanks to Branden Fitelson, David Ripley, Sven Rosenkranz and an anonymous referee for criticisms of an earlier draft of this paragraph.

[27]Obvious as this point may seem, its import has frequently been overlooked in certain debates involving the semantic paradoxes (see López de Sa and Zardini [2006]; [2007]; [2011]).

[28]Since we've reserved 'law of excluded middle' (i.e. 'LEM') for the logical claim expressed by '$\mathsf{t} \vdash \varphi \vee \neg\varphi$', let's use 'excluded middles' to refer to the typically non-logical claims expressed by instances of $\varphi \vee \neg\varphi$.

[29]I hasten to add that I don't mean to imply that a theory *must* solve all the paradoxes just mentioned in the text by deploying its fundamental thought, since, at least for the first two kinds of paradoxes, a theory might also appeal to *independently plausible truth-theoretic principles*, and, at least in the case of the truth-teller paradox, such appeal might *suffice to yield already a solution to the paradox* (for example, one might argue that the truth-teller simply lacks truth on the strength of general considerations concerning truth and grounding in reality, see Priest [2006], p. 66). Thanks to Patrick Greenough for urging this clarification.

One may try to address all these points by modifying slightly the proposal in question, saying that the fact that the theory rejects a sentence is actually a reflection of a more fundamental fact to the effect that the sentence is false *or somehow indeterminate*. A first problem with this modification is that, pending an *explanation* of the notion of indeterminacy at play, it is far from clear that it can effectively address the last two points of the previous paragraph. A second problem with the modification is that the relevant notion of indeterminacy supposed to apply to all paradoxical sentences is then typically explained in non-LEM theories *in terms of* LEM *failing for those sentences*, so that the alleged fundamental thought would finally collapse on the thought that LEM fails, which we've already seen does not in itself offer the materials for a solution to paradox B. A third problem with the modification is that it actually *still does not apply to all paradoxical sentences*: for example, the Epimenides sentence 'This sentence is not true and the number of stars in the universe is not even' is extremely plausibly false if the number of stars in the universe is even, and so there is no warrant to regard it as indeterminate in any reasonable sense. Concerning such a sentence, a non-LEM theorist could and should say that she rejects it and its negation because, *given what she knows*, it *might* be indeterminate. But since, given what she knows, that sentence also might be not indeterminate, to say so is in effect to concede that *there are paradoxical sentences which might be not indeterminate*. And since either our original Epimenides sentence or the opposite Epimenides sentence 'This sentence is not true and the number of stars in the universe is even' is false (for either the number of stars in the universe is even or it is not), to say so is in effect to concede that *there are paradoxical sentences which are not indeterminate*.[30] Moreover, rejection on such *epistemic* grounds no longer serves the purpose of addressing any of the points made in the previous paragraph which the modification discussed in this paragraph was supposed to address.[31]

[30]I've argued that there are paradoxical sentences which are not indeterminate, and that is enough to undermine the modification discussed in the text. It then becomes a secondary question whether the non-LEM theorist should say that LEM *fails* for whichever is the false sentence in the pair of opposite Epimenides sentences considered in the text. For what's worth, to me it would be very plausible to say that it does, and so concede that, in some cases, LEM fails even if a sentence is not indeterminate. But I suppose that one could instead say that it does not. To me, that would be very implausible. Obviously, since the sentence is false, the relevant excluded middle is true. But, since the sentence might have been indeterminate and cannot be known *a priori* not to be such (for the number of stars in the universe might have had the other parity and cannot be known *a priori* not to have it), to conclude from the truth of the relevant excluded middle that the relevant instance of LEM is valid is in effect to concede that, in some cases, *a sentence is a logical truth even if it is not necessarily such and even if it cannot be known a priori*, as well as to concede that, at least given transparency and in the broad sense explained in fn 2, *there is a logical proof* of the parity of the number of stars in the universe—in fact, a logical proof *of every actual truth*. Notice that exactly the same points apply to an Epimenides sentence *which we do know to be false*, such as 'This sentence is not true and there are no stars' (given which it would then become extremely natural to give the same treatment also to sentences which are necessarily false and known *a priori* (but not logically) to be false, such as 'This sentence is not true and something is not part of itself'). If so, the non-LEM theorist would have to concede for the Epimenides sentences she knows to be false that *LEM nevertheless fails for them* (and so that, at least in that sense, they are paradoxical) even if she believes them to be false, and so *even if she does not reject their negation*. Thanks to Timothy Williamson for discussion of this question.

[31]In view of this dialectic, one may try to take a rather different, more concrete approach focussing for example on a specific *model-theoretic construction* and saying that it is from the fundamental thought behind the construction that *both* denial of LEM *and* denial of the single-premise reduction theorem flow (although it should be mentioned that a non-LEM theorist in the spirit of Field [2008] would be reluctant to assign such a *fundamental explanatory role* to the model theory). A natural candidate for such a proposal is the

I've focussed so far on how well non-LEM theories fare with respect to paradox B. Non-LNC theories would block the paradox as presented already at subargument A_0, thus offering a unified solution to both paradox A and paradox B. If we try instead to run a *modified* version of paradox B using subargument A_0 only to get $T(\ulcorner\lambda\urcorner) \vdash \neg T(\ulcorner\lambda\urcorner)$, and then employ *reductio* to infer $\mathsf{t} \vdash \neg T(\ulcorner\lambda\urcorner)$ from that, we do obtain an argument that is valid by the lights of non-LNC theories, although, as I've noted at the beginning of this section, in the theoretic context of transparent theories the inference in question is, without further justification, very dubious at best (so much seems to be acknowledged by non-LNC theorists, since they typically feel the need to justify *reductio* and the immediately following claim that there are true contradictions by appealing to LEM along the lines of the argument I'll present at the end of this section; see e.g. Priest [2006], pp. 12–16, 64–66). But non-LNC theories would only accept the modified version of paradox B up to subargument B_1, and would deny the final subargument B_2 *qua* employing LNC, thus offering a unified solution to both paradox A and paradox B (original or modified).

However, we can also run a *dual modified* version of paradox B. In particular, given $\mathsf{t} \vdash \neg T(\ulcorner\lambda\urcorner)$, we can employ the metarule of the *single-conclusion demonstration theorem* (if $\mathsf{t} \vdash \varphi$ holds, $\neg\varphi \vdash \mathsf{f}$ holds) and infer $T(\ulcorner\lambda\urcorner) \vdash \mathsf{f}$ (typically, non-LNC theories, along with virtually all other theories, treat $\neg\neg\varphi$ as fully intersubstitutable with φ). Given that, by transparency, $\mathsf{t} \vdash \neg T(\ulcorner\lambda\urcorner)$ also implies $\mathsf{t} \vdash T(\ulcorner\lambda\urcorner)$, by transitivity we get $\mathsf{t} \vdash \mathsf{f}$.

---

strong-Kleene construction of Kripke [1975]. There is no doubt of course about the *technical* fact that the overall construction invalidates both LEM and the single-premise reduction theorem. What is open to doubt, however, is whether there is *a single fundamental thought* behind the overall construction. Let me explain. The *basic thought* behind the construction seems to be the "gaps-and-grounds" picture supporting the package constituted by the strong-Kleene valuation scheme together with the definition of truth at a later stage in terms of how things are at the earlier stage(s). Such basic thought thus includes the thought that *sentences might have a gappy status* (not to be identified with lack of truth and falsehood) which is a fixed point for negation and that is inherited by a disjunction from both of its disjuncts, a thought which, given plausible additional assumptions, is indeed sufficient to invalidate LEM. But such basic thought remains silent about the single-premise reduction theorem. That issue is only addressed by the *additional theoretic decision* of defining (single-premise, single-conclusion) logical consequence as *downwards preservation of the truth-entailing non-gappy status* (at the relevant fixed point(s)). But, precisely in the context of the basic thought in which *downwards preservation of the truth-entailing non-gappy status* does no longer coincide with *upwards preservation of the untruth-entailing non-gappy* status, that decision is arbitrary and indeed questionable in that it gives more importance to the truth-entailing non-gappy status than to the untruth-entailing one. The arbitrariness and indeed questionability of the decision may not be immediately apparent because one can still infer from the *truth of the premise* of a "valid" argument the *truth of the conclusion*, but it does emerge once it is noticed that one can no longer infer from the *untruth of the conclusion* of a "valid" argument the *untruth of the premise*. Once the alternative but more natural definition requiring *both* downwards preservation of the truth-entailing non-gappy status *and* upwards preservation of the untruth-entailing non-gappy status is adopted, the single-premise reduction theorem is validated. Moreover, the basic thought by itself already seems strongly to suggest the single-premise reduction theorem, and seems in any case to have consequences incompatible with non-LEM theories. For that thought also involves the thought that *a sentence is true in virtue of its positive grounding in reality*, from which it seems to follow that sentences that are not so grounded are not true (since they lack that in virtue of which a sentence is true). But, in the Kripke construction, if a sentence is a logical falsehood, it is not positively grounded in reality, and so it is not true. This strongly suggests the single-premise reduction theorem, and is in any case incompatible with non-LEM theories. Thanks to José Martínez, Sebastiano Moruzzi and Bryan Pickel for urging me to consider this alternative proposal.

A major feature of the dual modified version of paradox B is that it does not employ LNC. This is not to say that the dual modified version of paradox B has typically been taken to tell against non-LNC theories, for these theories, while accepting all the other principles employed in the dual modified version of paradox B, would deny the metarule of the single-conclusion demonstration theorem.

However, all the reasons given above in favour of the single-premise reduction theorem have dual reasons speaking in favour of the single-conclusion demonstration theorem (just like the single-premise reduction theorem infers logical truths from logical falsehoods, the single-conclusion demonstration theorem infers logical falsehoods from logical truths). If a sentence is a logical truth, it is true as a matter of logic and so its negation is false as a matter of logic and hence certainly a logical falsehood and thus entails f—in other words, just like we have the principle of *truth functionality* that [the negation of a sentence is false if the sentence is true], we also have the principle of *logical-truth functionality* that [the negation of a sentence is logically false if the sentence is logically true].

Using (only for the rest of this paragraph) 'inconsistent' instead of 'logical falsehood' in the same sense of and for the reasons explained eleven paragraphs back, the argument can be recast by saying that, if a sentence is a logical truth, it must$_L$ be true and so cannot$_L$ be untrue and hence, by the full power of transparency, its negation cannot$_L$ be true and thus certainly is inconsistent—the force of the argument remains strong and independent of our getting to attach the label 'logical falsehood' only to inconsistent sentences. I suppose that the non-LNC theorist will have to deny the inference from 'φ cannot$_L$ be true' to 'φ is inconsistent' just as she had to deny the inference from 'φ is a logical falsehood' to 'φ is inconsistent'.[32] The non-LNC theorist has thus to deny that *even the worst judgement* that modality$_L$ can pass on a sentence in itself suffices for the inconsistency of that sentence—indeed, it's easy to see that the non-LNC theorist even has to deny that it suffices for the mere lack of truth of that sentence (in this sense, although logical modality would still be powerful enough to establish many truths, somehow it would no longer be powerful enough to establish any lack of truth). Such a *lack of connection* between *logical consequence* (and even *mere lack of truth*) on the one hand and *negation* and *logical modality* on the other hand strikes me as a great cost of these theories.[33]

Obviously, the foregoing dialectic concerning the single-conclusion demonstration theorem could profitably be continued. Let's stop here however since, for the purposes of this paper, the point is not so much to argue against the non-LNC theorist's denial of the single-conclusion demonstration theorem, but only to establish the weaker point that the non-LNC theorist's

[32]The first two points made in fn 21 apply if she denies the duality of necessity$_L$ and possibility$_L$ (see also the similar argument in fn 33 that does not employ the duality of necessity$_L$ and possibility$_L$).

[33]Similarly, if a sentence is a logical truth, certainly it must$_L$ lack falsehood and so its negation must$_L$ lack truth and hence is inconsistent. I suppose that the non-LNC theorist will have to deny the inference from 'φ is a logical truth' to 'φ must$_L$ lack falsehood'. The non-LNC theorist has thus to deny that *even the best judgement* that logic can pass on a sentence in itself suffices for the necessary$_L$ lack of falsehood of that sentence—indeed, it's easy to see that the non-LNC theorist even has to deny that it suffices for the mere lack of falsehood of that sentence (in this sense, although logic would still be powerful enough to establish many truths, somehow it would no longer be powerful enough to establish any lack of falsehood). Such a *lack of connection* between *logical consequence* on the one hand and *lack of falsehood* and *logical modality* (and even *mere* lack of falsehood) on the other hand strikes me as a great cost of these theories.

solution to paradox B consisting in the denial of the single-conclusion demonstration theorem *does not flow* from her solution to paradox A consisting in the denial of LNC. The latter point can easily be established by noticing that denial of LNC is quite compatible with acceptance of the single-conclusion demonstration theorem: *dual intuitionist* logic provides a prime example of a coherent logical system lacking the former but having the latter (see e.g. Urbas [1996]). Thus, whatever rationale the non-LNC theorist might eventually come up with in order to justify her denial of the single-conclusion demonstration theorem, it cannot merely consist in her denial of LNC—in fact, our discussion of the single-conclusion demonstration theorem already amply shows that such a rationale would have to appeal to some rather unobvious considerations concerning the *lack of connection* between *logical consequence* on the one hand and *negation* and the accompanying notions of *(necessary$_L$) untruth* and *(logical) falsehood* on the other hand that are quite foreign to the issue as to whether LNC is valid.[34]

At this point, the non-LNC theorist might grant that denial of LNC does not offer a unified solution to the semantic paradoxes, but suggest that her unified solution comes rather from whatever turns out to be the fundamental thought behind her solution to the dual modified version of paradox B consisting in the denial of the single-conclusion demonstration theorem. Let's assume, plausibly, that such fundamental thought is the idea that *some sentences are such that both they and their negation are logical truths*. If that fundamental thought is correct, it provides a rationale for denying the single-conclusion demonstration theorem (and thus solving the dual modified version of paradox B): if both φ and $\neg\varphi$ are logical truths, the single-conclusion demonstration theorem implies the multiply repugnant conclusion that $\neg\varphi$ and $\neg\neg\varphi$ are logical falsehoods. The fundamental thought also provides a rationale for denying LNC (and thus solving paradox A): if both φ and $\neg\varphi$ are logical truths, by adjunction $\varphi \,\&\, \neg\varphi$ is also a logical truth, and so presumably not a logical falsehood.

Have we thus hit on a more appropriate formulation of non-LNC theories, one that allows for a unified solution to the semantic paradoxes? Emphatically no. Firstly, and less conclusively, the new fundamental thought is *not at all distinctive* of non-LNC theories: for example, *subvaluationist* and *non-standard revision* theories that do accept LNC and deny transparency would agree that some sentences are such that both they and their negation

[34]Another aspect of this lack of connection (strictly related to the principle of closure of (logical) falsehood under logical consequence discussed at the end of fn 19) concerns the traditional idea that *logical consequence consists in the impossibility$_L$ that [the premise is true and the conclusion is not true]*. Non-LEM theories need to reject that a premise entails a conclusion *only if* it is impossible$_L$ that [the premise is true and the conclusion is not true], since they accept that $T(\ulcorner\lambda\urcorner)$ entails f, and so would have to accept that it is impossible$_L$ (i.e. it is not possible$_L$) that [$T(\ulcorner\lambda\urcorner)$ is true and f is not true]. But, by the duality of necessity$_L$ and possibility$_L$, that implies that it is necessary$_L$ that it is not the case that [$T(\ulcorner\lambda\urcorner)$ is true and f is not true], and so, by the relevant De Morgan rule and closure of necessity$_L$ under entailment, it would be necessary$_L$ that either $T(\ulcorner\lambda\urcorner)$ is not true or f is true, and hence, reasoning by cases, by the properties of f and closure of necessity$_L$ under entailment, it would be necessary$_L$ that $T(\ulcorner\lambda\urcorner)$ is not true, which is however unacceptable for non-LEM theories. Non-LNC theories need to reject that a premise entails a conclusion *if* it is impossible$_L$ that [the premise is true and the conclusion is not true], since they accept that it is necessary$_L$ that λ is not true. By the contrapositive of addition and closure of necessity$_L$ under entailment, that implies that it is necessary$_L$ that it is not the case that [λ is true and f is not true], and so, by the duality of necessity$_L$ and possibility$_L$, that it is not possible$_L$ (i.e. it is impossible$_L$) that [λ is true and f is not true], and hence non-LNC theories would have to accept that λ entails f, which is however unacceptable for them.

are logical truths.[35,36] On the conception in question, the *distinctiveness* of non-LNC theories against all the other main alternative theories would not consist in what would be the *fundamental thought* affording them a unified solution to the semantic paradoxes (i.e. the thought that some sentences are such that both they and their negation are logical truths); it would rather consist in (that thought plus) an *ancillary thought* about conjunction (i.e. the thought that LNC fails for all sentences such that both they and their negation are logical truths). The two thoughts need not of course be completely unrelated—I've mentioned in the previous paragraph how the fundamental thought can be used to yield the ancillary one given plausible additional assumptions about conjunction.[37] But the sore point remains that, on the conception in question, the fundamental thought affording to a non-LNC theory a unified solution to the semantic paradoxes is divorced from the thought characterising it against all the other main alternative theories. Moreover, as we'll appreciate over the next few paragraphs, this problem has the tendency to degenerate badly—especially with respect to the connection between the fundamental thought and the ancillary thought—once it interacts with another somewhat converse problem to which we now turn.

Secondly, and more conclusively, not every semantic paradox has truck with even apparent logical truths: the same examples discussed in connection with non-LEM theories are relevant here as well. The natural solution given to these paradoxes by non-LNC theories (with the exception of the relevant versions of Epimenides' paradox) consists in accepting the relevant contradictions (and hence denying LNC), rather than in claiming that the relevant sentences are such that both they and their negation are logical truths.

In response, one could, I suppose, stretch the notion of "logic" at play in a non-LNC theory so that every sentence accepted by the theory counts as a "logical truth". But such a proposal would be unsatisfactory for various related reasons. Firstly, in the new stretched sense of 'logical truth', the apparently *objective* claim that some sentences are such that both they and their negation are "logical truths" boils down to the *autobiographical* claim that some sentences are such that both they and their negation are accepted by the theory (the theorist really), which can hardly be taken to be a fundamental thought behind a solution to the semantic paradoxes. In fact, such a claim immediately cries out for a deeper explanation: why does the theory accept both a sentence and its negation? Secondly, the proposal does

[35] To the best of my knowledge, such theories have not been investigated in relation to the semantic paradoxes. I won't go into their details in this paper—suffice it to say that they naturally arise by dualising, respectively, the familiar Kripke construction based on the supervaluationist evaluation scheme and the familiar revision-sequence construction. Such theories are very similar to one another in the respects that are relevant for our discussion: in particular, they accept both a sentence and its negation without accepting any contradiction (more strongly, they hold that the contradiction is a logical falsehood and accept LNC in its full generality). The underlying idea, to put it very roughly, is that conjunction is sensitive to *compatibility relationships* between the conjuncts.

[36] A comment analogous to that in fn 23 applies concerning the *complete indistinguishability* of all these theories in expressively impoverished paradoxical languages.

[37] Such assumptions start to look less plausible *vis-à-vis* the alternative assumptions made by subvaluationist and non-standard revision theories once it is realised that, keeping fixed the duality of conjunction and disjunction and the idea that some sentences are such that both they and their negation are logical truths, the latter assumptions allow subvaluationist and non-standard revision theories to uphold the very plausible claim that the negation of any excluded middle is a logical falsehood, while the former assumptions force non-LNC theories to endorse the very implausible claim that the negations of certain excluded middles are logical truths.

nothing but *exacerbate* the problem discussed two paragraphs back of distinguishing the fundamental thought of non-LNC theories from what the other main alternative theories endorse: *virtually all theories* that accept the equivalence between $T(\ulcorner\varphi\urcorner)$ and φ but do not reject both a sentence and its negation agree in accepting the relevant paradoxical sentences and their negation. Thirdly, the ancillary thought about conjunction concerning the failure of LNC *does no longer even remotely follow from the fundamental thought as so weakly understood*: sometimes, accepting both conjuncts is no sufficient ground for accepting a conjunction (as evidenced by the *preface paradox*, see Makinson [1965]).

One may try to address all these points by modifying slightly the proposal in question, saying that the fact that the theory accepts a sentence is actually a reflection of a more fundamental fact to the effect that the sentence is true-only *or somehow overdeterminate*. A first problem with this modification is that, pending an *explanation* of the notion of overdeterminacy at play, it is far from clear that it can effectively address the last two points of the previous paragraph. A second problem with the modification is that the relevant notion of overdeterminacy supposed to apply to all paradoxical sentences is then typically explained in non-LNC theories *in terms of* LNC *failing for those sentences*, so that the alleged fundamental thought would finally collapse on the thought that LNC fails, which we've already seen does not in itself offer the materials for a solution to the dual modified version of paradox B. A third problem with the modification is that it actually *still does not apply to all paradoxical sentences*: for example, the Epimenides sentence 'This sentence is not true and the number of stars in the universe is not even' is extremely plausibly false-only if the number of stars in the universe is even, and so there is no warrant to regard it as overdeterminate in any reasonable sense. Rather implausibly, concerning such a sentence a non-LNC theorist could say that she *accepts* it because, *given what she knows*, it *might* be overdeterminate. But since, given what she knows, that sentence also might be not overdeterminate, to say so is in effect to concede that *there are paradoxical sentences which might be not overdeterminate*. And since either our original Epimenides sentence or the opposite Epimenides sentence 'This sentence is not true and the number of stars in the universe is even' is false-only (for either the number of stars in the universe is even or it is not), to say so is in effect to concede that *there are paradoxical sentences which are not overdeterminate*.[38] Moreover, acceptance on such *epistemic* grounds no longer serves the purpose of addressing any of the points made in the previous paragraph which the modification discussed in this paragraph was supposed to address. More plausibly, concerning both such sentences a non-LNC theorist could say that she *rejects* them because, *given what she knows*, either *might* be false-only. But, if she does so, there would be paradoxical sentences that are not such that both they and their negation are accepted by the theory, contrary to what the proposal in question requires.[39]

[38]Comments analogous to those in fn 30 apply concerning the question whether the non-LNC theorist should say that LNC fails for whichever is the false-only sentence in the pair of opposite Epimenides sentences considered in the text.

[39]Comments analogous to those in fn 31 apply concerning a rather different, more concrete approach focussing for example on a specific *model-theoretic* construction and saying that it is from the fundamental thought behind the construction that *both* denial of LNC and denial of the single-conclusion demonstration theorem flow. In particular, the first comment in fn 31 has an analogue to the effect that the decision of defining (single-premise, single-conclusion) logical consequence as *downwards preservation of truth-entailing non-glutty or glutty status* (at the relevant fixed point(s)), without requiring *upwards preservation of untruth-entailing non-glutty or glutty status*, is arbitrary and indeed questionable. The second comment in fn 31 has

In stark contrast with the severe difficulties for non-LEM and non-LNC theories in giving a unified solution to paradoxes A and B (original or modified or dual modified), **IKT** offers a smooth treatment of both: that theory denies subargument A_0 as it employs contraction, thus blocking paradox A and the original version of paradox B; moreover, the theory does accept subargument A_0 up until $T(\ulcorner\lambda\urcorner) \vdash \neg T(\ulcorner\lambda\urcorner)$, but denies *reductio*, thus blocking the modified and dual modified versions of paradox B. This is in the ball-park for being a unified solution because, contrary to the single-premise reduction theorem, I've argued above that, in the theoretic context of transparent theories, *reductio* is in the "better" case (i.e. if there are true contradictions), without further justification, very dubious at best while it is in the "worse" case (i.e. if there are no true contradictions) straightforwardly incompatible with the impredicative phenomena that go together with transparency, and there is no obligation for a unified transparent solution to deploy its fundamental thought in order to deny an argument that relies, without further justification, on such a problematic principle.[40]

Still, albeit in these respects extremely problematic, *reductio* can actually be further justified by a couple of apparently compelling arguments, so that—supplementing the modified or dual modified version of paradox B with some such argument—we do have a genuine paradox to block. The first argument corresponds to the first understanding of *reductio* mentioned in fn 17, and is most naturally applied to its intuitionistically acceptable version. The argument assumes that a formula φ entails its own negation and reasons as follows:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize reflexivity}
\UnaryInfC{$\varphi \vdash \varphi$}
\AxiomC{$\varphi \vdash \neg\varphi$}
\RightLabel{\scriptsize adjunction}
\BinaryInfC{$\varphi, \varphi \vdash \varphi \,\&\, \neg\varphi$}
\RightLabel{\scriptsize contraction}
\UnaryInfC{$\varphi \vdash \varphi \,\&\, \neg\varphi$}
\AxiomC{}
\RightLabel{\scriptsize LNC}
\UnaryInfC{$\varphi \,\&\, \neg\varphi \vdash \mathsf{f}$}
\RightLabel{\scriptsize transitivity}
\BinaryInfC{$\varphi \vdash \mathsf{f}$}
\RightLabel{\scriptsize single-premise reduction theorem}
\UnaryInfC{$\mathsf{t} \vdash \neg\varphi$}
\end{prooftree}
$$

As we've seen, non-LEM theories block this argument by denying the apparently compelling single-premise reduction theorem, getting involved in the dialectic examined above concerning the original version of paradox B; non-LNC theories, while not accepting this particular

an analogue to the effect that the basic thought behind the relevant construction involves the thought that *a sentence is not true in virtue of its negative grounding in reality*, from which it seems to follow that sentences that are not so grounded lack untruth (since they lack that in virtue of which a sentence is untrue).

[40] Compare the similar argument consisting only in the inference from $T(\ulcorner\lambda\urcorner) \vdash \neg T(\ulcorner\lambda\urcorner)$ and $\neg T(\ulcorner\lambda\urcorner) \vdash T(\ulcorner\lambda\urcorner)$ to $\mathsf{t} \vdash$ 'Classical mathematics is inconsistent'. Although that argument is classically valid, there is no obligation for a unified transparent solution to deploy its fundamental thought in order to deny the only inference employed in the argument—in the theoretic context of transparent theories, that inference is, without further justification, very dubious at best. This is not to deny of course that the inference might be *further justified* by appeal to *more fundamental and, even in the theoretic context of transparent theories, apparently compelling principles*, so as to produce a *genuine paradox* for those theories that should ideally be blocked by deploying their fundamental thought. That is in fact what I've done for the single-premise reduction theorem and the single-conclusion demonstration theorem (with the main aim of showing that non-LEM and non-LNC theories need to appeal to some rather unobvious considerations concerning the lack of connection between logical consequence and negation that are quite foreign to the issue as to whether LEM or LNC are valid), and what I'll proceed to do for *reductio* (with the main aim of showing that the justifying arguments do involve contraction). Thanks to Sven Rosenkranz and an anonymous referee for criticisms of an earlier draft of this paragraph.

argument, accept *reductio* anyways (as I've already mentioned, typically feeling the need to justify it and the immediately following claim that there are true contradictions by appealing to LEM along the lines of the argument I'll present in the next paragraph), getting rather involved in the dialectic examined above concerning the dual modified version of paradox B. **IKT**, on the contrary, denies this argument as it employs contraction, thus blocking it (and the original version of paradox B) *in exactly the same way* as paradox A.

The second argument corresponds to the second understanding of *reductio* mentioned in fn 17, and is most naturally applied to its distinctively classical version. The argument assumes that a formula φ follows from its own negation and reasons as follows:

$$\dfrac{\dfrac{}{\mathsf{t} \vdash \varphi \lor \neg\varphi}\,\text{LEM} \qquad \dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}\,\text{reflexivity} \qquad \neg\varphi \vdash \varphi}{\varphi\lor\neg\varphi \vdash \varphi,\varphi}\,\text{abjunction}}{\varphi\lor\neg\varphi\vdash\varphi}\,\text{contraction}}{\mathsf{t}\vdash\varphi}\,\text{transitivity}$$

Non-LEM theories block this argument by denying LEM; non-LNC theories accept this argument and, as I've already mentioned, typically use it to justify *reductio* and the immediately following claim that there are true contradictions, getting involved in the dialectic examined above concerning the dual modified version of paradox B. **IKT**, on the contrary, denies this argument as it employs contraction, thus blocking it (and the original version of paradox B) *in exactly the same way* as paradox A.

Since **IKT** blocks all of paradox A, the original version, the most compelling modified versions and the most compelling dual modified versions of paradox B in the same way, it does offer a *unified* solution to these paradoxes—what non-LEM and non-LNC theories fail to do. Notice that the solution is also *not overdetermined*: all the other principles employed in paradox A and in the original version, the most compelling modified versions and the most compelling dual modified versions of paradox B are valid according to **IKT**—in particular, LEM, LNC, the single-premise reduction theorem and the single-conclusion demonstration theorem are all valid according to **IKT**.

4 A Unified Solution to the Liar and Curry's Paradox

Although the semantic paradoxes historically emerge with the Liar paradox in some of its versions, as a matter of autobiographical remark the original and constantly guiding source of inspiration for **IKT** has been a certain version of *Curry's* paradox (Curry [1942] is the modern *locus classicus* while Ashworth [1974], p. 125 mentions some prominent scholastic antecedents). So let's examine in some detail the workings of that version, considering a sentence κ identical to $T(\ulcorner\kappa\urcorner) \rightarrow \bot$ (where $\bot$ is the conjunction of all propositions). We start with:

$$\dfrac{\dfrac{\dfrac{}{T(\ulcorner\kappa\urcorner) \rightarrow \bot, T(\ulcorner\kappa\urcorner) \vdash \bot}\,\textit{modus ponens}}{T(\ulcorner\kappa\urcorner), T(\ulcorner\kappa\urcorner) \vdash \bot}\,\text{transparency}}{T(\ulcorner\kappa\urcorner) \vdash \bot}\,\text{contraction}$$

(call this reasoning 'C_0'). We continue with:

$$\dfrac{\dfrac{}{T(\ulcorner\kappa\urcorner)\ \vdash\ \bot}\,\mathrm{C}_0}{\dfrac{\mathsf{t}\ \vdash\ T(\ulcorner\kappa\urcorner)\rightarrow\bot}{\mathsf{t}\ \vdash\ T(\ulcorner\kappa\urcorner)}\,\text{transparency}}\,\text{single-premise deduction theorem}$$

(call this reasoning 'C_1'). We close with:

$$\dfrac{\dfrac{}{\mathsf{t}\ \vdash\ T(\ulcorner\kappa\urcorner)}\,\mathrm{C}_1 \qquad \dfrac{}{T(\ulcorner\kappa\urcorner)\ \vdash\ \bot}\,\mathrm{C}_0}{\mathsf{t}\ \vdash\ \bot}\,\text{transitivity}$$

(call this reasoning 'C_2', and the whole argument 'paradox C').

Like paradox B, a major feature of paradox C is that it employs neither LEM nor LNC. This is not to say that paradox C has typically been taken to tell against non-LEM or non-LNC theories, for these theories, while accepting all the other principles employed in paradox C, would typically deny the metarule of the *single-premise deduction theorem* (if $\varphi \vdash \psi$ holds, $\mathsf{t} \vdash \varphi \rightarrow \psi$ holds). A discussion of the single-premise deduction theorem (and of its relationship to *absorption*) is thus in order. Consider first absorption. This comes at least in two versions (both of which are classically valid): a well-known *law* version ($\mathsf{t} \vdash \varphi \rightarrow (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$) and a less well-known *metarule* version (if $\Gamma, \varphi \vdash \Delta, \varphi \rightarrow \psi$ holds, $\Gamma \vdash \Delta, \varphi \rightarrow \psi$ holds).[41] Now, any transparent theory generates so much *impredicativity* as to yield *ungrounded sentences that are strongly equivalent with their own conditional to an unacceptable sentence*—sentences that in some sense consist in their self-conditional to an unacceptable sentence (κ being a standard example of this). It is for this reason that, in the theoretic context of transparent theories, the law and the metarule of absorption immediately look as rather silly principles: in that context, it becomes one of the most distorting features of classical logic that *it rules out the very existence of such sentences*, and, given closure of logical truth under *modus ponens*, the law and the metarule of absorption are already sufficient to yield that most distorting result.

Let's now move on to consider the deduction theorem. Consider first the *multiple-premise deduction theorem* (if $\Gamma, \varphi \vdash \psi$ holds, $\Gamma \vdash \varphi \rightarrow \psi$ holds). Now, any theory that both accepts *monotonicity* and aims at preserving *relevance* for $\rightarrow$ should deny the multiple-premise deduction theorem. For, by reflexivity, $\varphi \vdash \varphi$ holds, and so, by monotonicity, $\varphi, \psi \vdash \varphi$ holds, and hence, by the multiple-premise deduction theorem, $\varphi \vdash \psi \rightarrow \varphi$ holds, thus yielding the "positive paradox of material implication". Interesting as it may be in other contexts, this point against the multiple-premise deduction theorem is far from being conclusive in ours, as many non-LEM and non-LNC theories do not accept relevance constraints on $\rightarrow$. Nevertheless, non-LEM and non-LNC theories unfortunately turn out to be banned on other

[41]Having officially recorded the version with side-premises and side-conclusions, for simplicity I'll ignore these in my treatment of the metarule of absorption.

grounds from accepting close kins of the single-premise deduction theorem (and turn out to be so banned independently of the semantic paradoxes). As for non-LEM theories, by reflexivity, $\varphi \vdash \varphi$ holds, and so, by monotonicity, $\varphi \vdash \varphi, \bot$ holds, and hence, by the *multiple-conclusion deduction theorem* (if $\varphi \vdash \Delta, \psi$ holds, $\mathsf{t} \vdash \Delta, \varphi \to \psi$ holds), $\mathsf{t} \vdash \varphi, \varphi \to \bot$ holds. Since in non-LEM theories $\varphi \to \bot \vdash \neg\varphi$ typically holds, by transitivity $\mathsf{t} \vdash \varphi, \neg\varphi$ holds, and so, by disjunction in the conclusions, LEM follows. Since non-LEM theories typically accept all the other principles employed in this reasoning, they must deny the multiple-conclusion deduction theorem. A dual point can be made for non-LNC theories by introducing the dual of the conditional, the *unconditional* $\dot{-}$ (read $\varphi \dot{-} \psi$ informally as something like 'Its being the case that φ does not require its being the case that ψ'). By reflexivity, $\varphi \vdash \varphi$ holds, and so, by monotonicity, $\varphi, \top \vdash \varphi$ holds (where $\top$ is the disjunction of all propositions), and hence, by the *multiple-premise deduction theorem for the unconditional* (if $\Gamma, \varphi \vdash \psi$ holds, $\Gamma, \varphi \dot{-} \psi \vdash \mathsf{f}$ holds), $\varphi, \top \dot{-} \varphi \vdash \mathsf{f}$ holds. Since in non-LNC theories $\neg\varphi \vdash \top \dot{-} \varphi$ should presumably hold, by transitivity $\varphi, \neg\varphi \vdash \mathsf{f}$ holds, and so, by conjunction in the premises, LNC follows. Since non-LNC theories typically accept or should presumably accept all the other principles employed in this reasoning, they must deny the multiple-premise deduction theorem for the unconditional.

Let's grant for the sake of argument that non-LEM and non-LNC theories can make good sense of their denial of close kins of the single-premise deduction theorem; after all, as I've already mentioned, the problems raised in the previous paragraph do not rely on any truth-theoretic principle, and so they are much more general problems affecting virtually all theories that deny LEM or LNC (such as intuitionist logic and dual intuitionist logic respectively). The problem becomes much more specific with the single-premise deduction theorem. Contrary to the law and the metarule of absorption and to the multiple-conclusion deduction theorem, the input of the single-premise deduction theorem is not that a sentence implies or entails its own conditional to a sentence or that it entails a sentence as a side-conclusion—it is the intuitively much stronger one that a sentence entails a sentence, and so that the former sentence logically implies the latter sentence. Even on a transparent theory, the single-premise deduction theorem would seem irresistible: if a sentence logically implies a sentence, certainly the former sentence implies the latter sentence as a matter of logic, and so the conditional is true as a matter of logic and hence a logical truth. The bad news for non-LEM and non-LNC theories is that, compelling as these considerations may seem, they ought to be rejected by a non-LEM or non-LNC theory on pain of paradox C.

There is an apparently terminological issue with the way the argument in the previous paragraph in favour of the single-premise deduction theorem has been presented which may be worth addressing. The argument plausibly assumes that *cases of entailment* are also *cases of logical implication*. The apparently terminological issue concerns the likely protest of a non-LEM or non-LNC theorist to the effect that the epithet 'logical implication' should be reserved for *cases of logically true conditionals*. In fact, while the two properties are co-extensional in classical logic and in **IKT**, they come apart in both non-LEM and non-LNC theories (in particular, in these theories, although every case of a logically true conditional is a case of entailment, some cases of entailment are not cases of logically true conditionals). Let's grant (only for the rest of this paragraph) this apparently terminological point about 'logical implication'.[42] That does not improve much the situation for non-LEM and non-LNC theories, for the argument in the previous paragraph in favour of the single-premise deduction theorem can be recast by saying that, if a sentence entails a sentence, certainly the latter sentence must$_L$ be true if the former sentence is true, and so, by the full power of transparency, the conditional from the former to the latter must$_L$ be true and thus is a logical truth—the force of the argument remains strong and independent of our getting to attach the label 'logical implication' to cases of entailment. I suppose that the non-LEM or non-LNC theorist will have to deny the inference from 'φ entails ψ' to 'ψ must$_L$ be true if φ is true' just as she had to deny the inference from 'φ entails ψ' to 'φ logically implies ψ'. The non-LEM or non-LNC theorist has thus to deny that *even the best judgement* that logic can pass on one sentence in itself with respect to another sentence in itself suffices for the necessary$_L$ truth preservation from the latter sentence to the former sentence—indeed, it's easy to see that the non-LEM or non-LNC theorist even has to deny that it suffices for the mere truth preservation from the latter to the former (in this sense, although logic would still be powerful enough to establish many truths, somehow it would no longer be powerful enough to establish any preservation of truth). Such a *lack of connection* between *logical consequence* on the one hand and the *conditional* and *logical modality* (and even the *mere* conditional) on the other hand strikes me as a great cost of these theories.

Obviously, the foregoing dialectic concerning the single-premise deduction theorem could profitably be continued. Let's stop here however since, for the purposes of this paper, the point is not so much to argue against the non-LEM or non-LNC theorist's denial of the single-premise deduction theorem, but only to establish the weaker point that the non-LEM or non-LNC theorist's solution to paradox C consisting in the denial of the single-premise deduction theorem *does not flow* from her solution to paradox A consisting in the denial of LEM or LNC. The latter point can easily be established by noticing that denial of LEM or LNC is quite compatible with acceptance of the single-premise deduction theorem: the latter principle—and paradox C more generally—concerns implication and does not concern negation at all, while the latter principles concern negation and do not concern implication at all. Thus, whatever rationale the non-LEM or non-LNC theorist might eventually come up with in order to justify her denial of the single-premise deduction theorem, it cannot merely consist in her denial of LEM or LNC—in fact, our discussion of the single-premise deduction theorem already amply shows that such a rationale would have to appeal to some rather unobvious considerations concerning the *lack of connection* between *logical consequence* on the one hand and the *conditional* and the accompanying notions of *(necessary$_L$) truth preservation* and

[42] Comments analogous to the first two in fn 19 apply concerning the only apparent terminological character of the point. In particular, the second comment in fn 19 has an analogue to the effect that, just as the best judgement that logic can pass on a sentence in itself—that it is the conclusion of a no-premise, single-conclusion argument—is equal to a judgement of validity for the categorical statement consisting in that sentence, so *the best judgement that logic can pass on one sentence in itself with respect to another sentence in itself*—i.e. that it is the conclusion of a single-premise, single-conclusion argument whose premise is the latter sentence—*should be equal to a judgement of validity for the hypothetical statement from the latter sentence to the former sentence*—i.e. to a logical implication from the latter to the former. A third argument for the conclusion that logical implication just is entailment assumes very plausibly that entailment, usually presented as a *metalinguistic* relation, is ultimately just a kind of *object-linguistic* implication (lying at one extreme of a spectrum at whose other extreme lies material implication), which should then be identified with logical implication.

(logical) implication on the other hand that are quite foreign to the issue as to whether LEM or LNC is valid—and, more generally, quite foreign to the further issues involving negation discussed in section 3.[43]

In fact, in a way sharper than the one exemplified by the Epimenides sentences discussed in section 3 (especially those discussed at the end of fn 30), a Curry sentence like κ is an example of a full-blooded paradoxical sentence for which the non-LNC theorist *cannot possibly accept the corresponding contradiction* (and so *cannot possibly envisage a failure of LNC*), as she *cannot possibly accept* κ in the first place (on pain of having to accept $\perp$ by contraction, transparency and *modus ponens*). This point is already well-known even if not often stressed (see Field [2008], pp. 380–381 for a recent restatement). What bears emphasis is that a dual point affects non-LEM theories just as well. Considering a sentence ι identical to $\top \dot{-} T(\ulcorner\iota\urcorner)$ and employing *modus ponens for the unconditional* ($\varphi \vdash \psi, \varphi \dot{-} \psi$) and the *single-premise deduction theorem for the unconditional* (if $\varphi \vdash \psi$ holds, $\varphi \dot{-} \psi \vdash \mathsf{f}$ holds), a dual version of paradox C leads to the conclusions that $\top \vdash T(\ulcorner\iota\urcorner)$, $\top \dot{-} T(\ulcorner\iota\urcorner) \vdash \mathsf{f}$, $T(\ulcorner\iota\urcorner) \vdash \mathsf{f}$ and $\top \vdash \mathsf{f}$ hold (I assume that, at least for some unconditional, non-LEM theorists will accept *modus ponens* for it and deny the single-premise deduction theorem for it, thus accepting only the first consequence mentioned). Thus, a dual Curry sentence like ι is an example of a full-blooded paradoxical sentence for which the non-LEM theorist *cannot possibly reject the corresponding excluded middle* (and so *cannot possibly envisage a failure of LEM*), as she *cannot possibly reject* ι in the first place (on pain of having to reject $\top$ by contraction, transparency and *modus ponens* for the unconditional).

In stark contrast with the severe difficulties for non-LEM and non-LNC theories in giving a unified solution to paradoxes A and C, **IKT** offers a smooth treatment of both: the theory denies subargument C_0 (and its dual version) as it employs contraction, thus blocking paradox C (and its dual version) *in exactly the same way* as paradox A. Since **IKT** blocks all of paradox A, the original version, the most compelling modified versions, the most compelling dual modified versions of paradox B and the original version and the dual version of paradox C

[43] The point is in effect conceded by a prominent non-LNC theorist who has often emphasised the importance of offering a unified solution to the semantic paradoxes: "[...] the curried versions of the paradoxes belong to a quite different family. Such paradoxes do not involve negation and, *a fortiori*, contradiction", "They are paradoxes that involve essentially conditionality [...] Genuine Curry paradoxes are therefore ones that depend on a mistaken theory of the conditional, and are perhaps best thought of as more like the 'paradoxes of material implication'" (Priest [2003], pp. 169, 278). In fact, the claim could be made that the fundamental version of Curry's paradox does not satisfy the *inclosure schema* that Priest [2003] has argued to be at the root of the semantic paradoxes. Although a proper treatment of this issue lies beyond the scope of this paper, I should record that, if that claim were correct, it would seem to me more plausible to take it to reflect badly on the inclosure schema as a diagnosis of the semantic paradoxes rather than on Curry's paradox as a semantic paradox. It may also be worth mentioning that a non-LEM or non-LNC theory might be such that, if the relevant instances of LEM or LNC are added for a certain fragment of the language, that fragment "behaves classically", and so in particular obeys the single-premise deduction theorem (the theory in Field [2008] is an example of such a system). Even this (possible) technical fact would however be very far from indicating that, in some reasonable sense, the single-premise deduction theorem fails in such a theory *because* LEM or LNC fail in the theory. Compare: the technical fact about intuitionist logic that, if the relevant instances of *Peirce's law* ($\mathsf{t} \vdash ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$) are added for a certain fragment of the language, that fragment "behaves classically", and so in particular obeys LEM, is very far from indicating that, in some reasonable sense, LEM fails in intuitionist logic *because* Peirce's law fails in the logic. Thanks to Graham Priest for putting forth to me the claim about the inclosure schema mentioned in this fn.

in the same way, it offers a *unified* solution to these paradoxes—what non-LEM and non-LNC theories fail to do. Notice that the solution is also *not overdetermined*: all the other principles employed in paradox A, in the original version, the most compelling modified versions, the most compelling dual modified versions of paradox B and in the original version and the dual version of paradox C are valid according to **IKT**—in particular, LEM, LNC, the single-premise reduction theorem, the single-conclusion demonstration theorem, the single-premise deduction theorem and the single-premise deduction theorem for the unconditional (as well as *modus ponens* and *modus ponens* for the unconditional) are all valid according to **IKT**.

The contrast between the unified solution offered by **IKT** and the non-unified solutions offered by non-LEM and non-LNC theories is but amplified if we consider for example the denial of the law of absorption (in terms of which Curry [1942] originally presented Curry's paradox). I've explained above how, in the theoretic context of transparent theories, that principle is extremely problematic. One could try to justify the principle by saying that if φ implies $\varphi \rightarrow \psi$, since it also implies φ, it should imply ψ, as it implies both premises of *modus ponens*. This justification would not have much force for non-LEM and non-LNC theories, since, for better or worse, on these theories the deduction theorem has to fail in such a dramatic way that, although *modus ponens* is valid, its two premises do not imply its conclusion. But one could also justify the law of absorption by saying that, *under the supposition that P, nothing new comes up if it is "further" supposed that P*, so that if, under the supposition that P, if it is "further" supposed that P, it results that Q, then, under the supposition that P, it already results that Q. The law of absorption follows. The argument must presumably be blocked at the assumption that, under the supposition that P, nothing new comes up if it is "further" supposed that P. But, while that assumption is indeed extremely problematic in the absence of contraction, it has nothing to do with LEM, LNC or the single-premise deduction theorem (in fact, sometimes in conversation sympathisers of non-LEM or non-LNC theories have incautiously made fun of **IKT** precisely because of its denial of that and similar assumptions). Not only cannot non-LEM and non-LNC theories offer a unified solution to the Liar and Curry's paradox; they cannot even offer a unified solution to Curry's paradox.

I'd like to close this discussion by making more explicit an underlying theme of this paper. It was relatively easy to make the main point of this section, as it is pretty clear that paradox C requires non-LEM and non-LNC theories to revise the logic of the *conditional* in (implausible) ways that go beyond what is required by denial of LEM or LNC (in particular, in ways that prevent facts about logical consequence from having the expected effects at the level of facts about the conditional). The ambitious task was rather the one undertaken in section 3, to the effect that, in an analogous fashion, already a variation on paradox A like paradox B requires non-LEM and non-LNC theories to revise the logic of *negation* in (implausible) ways that go beyond what is required by denial of LEM or LNC (in particular, in ways that prevent facts about logical consequence from having the expected effects at the level of facts about negation). Thus, it is true that, with respect to paradox A, paradox C reveals new conceptual difficulties for non-LEM and non-LNC theories in the treatment of the conditional, but reflection on paradox B reveals that *analogous difficulties were already present in the treatment of negation*. Indeed, paradox C suggests a *final* version of paradox B which replaces κ with λ, $T(\ulcorner\kappa\urcorner)$ with $T(\ulcorner\lambda\urcorner)$, $\perp$ with f, *modus ponens* with the law of *exclusion* ($\varphi, \neg\varphi \vdash \mathsf{f}$) and the single-premise deduction theorem with the single-premise reduction theorem. I regard this final version of paradox B as one of the most challenging semantic paradoxes for non-LEM, non-LNC and, more generally, non-substructural theories: just as paradox C shows that, very surprisingly, in these theories no conditional that is *weak enough to record logical implication* can still be *strong enough to licence the inference from its antecedent to its consequent* (two jobs that, far for from being in tension, seem to cohere very well with each other), so the final version of paradox B shows that, very surprisingly, in these theories no negation that is *weak enough to record logical falsehood* can still be *strong enough to exclude its negatum* (again, two jobs that, far for from being in tension, seem to cohere very well with each other).

5 Getting One for Two

The semantic paradoxes obviously come in many more kinds than the Liar or Curry's paradox, and it is an incumbent task to show how the unified non-contractive solution to the latter two paradoxes presented here can be extended to cover all semantic paradoxes. Such task lies however beyond the scope of this paper—it was here sufficient to show that **IKT** does offer a unified solution to the two most prominent kinds of semantic paradoxes. Such solution is in a sense simple: all the paradoxes reviewed in this paper commit the fallacy of inferring from the fact that a certain premise *taken twice* entails a certain conclusion that the premise *taken once* still entails the conclusion (or the corresponding fallacy concerning contraction in the conclusions). They commit the fallacy of *contracting* two occurrences of a premise or conclusion into one. The fallacy is thus *purely structural*: it does not have anything to do with *specific logical operations* like negation or the conditional. In particular, it does not have anything to do with negation. In fact, in **IKT** negation is in a very good sense *completely classical*, since it obeys the characteristically Boolean principles $\neg$-L and $\neg$-R: $\neg\varphi$ is the sentence that holds in all and only those cases in which φ fails to hold. Non-LEM and non-LNC theories suppose otherwise, and assume that rejecting either of these features of Boolean negation is the key to the semantic paradoxes. However, we've seen that such rejection is neither here nor there even for some versions of the Liar paradox, let alone for Curry's paradox.

But why is getting one for two a fallacy? Why does contraction fail? My own view—which I have to some extent developed in Zardini [2011], pp. 503–506 and of which I can only offer the merest sketch here—is that it fails because the relevant sentences express *unstable states-of-affairs*, i.e. states-of-affairs that *lead to consequences with which they need not co-obtain*.[44] If φ expresses the state-of-affairs s_0, $\varphi, \varphi \vdash \psi$ may hold, let's suppose, only because s_0 and some state-of-affairs s_1 consequence of s_0 *together* directly lead to the state-of-affairs s_2 expressed by ψ. If s_0 is however unstable, it does not follow that s_0 *by itself* leads to s_2, and so it does not follow that $\varphi \vdash \psi$ holds. For, although s_0 does of course by itself lead to its consequence s_1, by its instability s_0 *need not co-obtain with* s_1, while we're supposing that s_0 can lead to s_2 *only together with* s_1. Failure of contraction is thus the logical symptom of an underlying unstable metaphysical reality. The investigations in this paper invite then the conjecture that

[44] As usual, states-of-affairs are abstract entities that can either obtain or fail to obtain. A locution like 'State-of-affairs x leads to state-of-affairs y' must be understood as 'The *obtaining* of state-of-affairs x leads to the *obtaining* of state-of-affairs y'.

it is precisely this unstable reality that is at the root of the semantic paradoxes.

References

Earline Ashworth. *Language and Logic in the Post-Medieval Period.* Reidel, Dordrecht, 1974.

JC Beall. *Spandrels of Truth.* Oxford University Press, Oxford, 2009.

Jozef Bocheński. *A History of Formal Logic.* Chelsea, New York, 2nd edition, 1970.

Ross Brady. *Universal Logic.* CSLI Publications, Stanford, 2006.

Haskell Curry. The inconsistency of certain formal logics. *The Journal of Symbolic Logic*, 7: 115–117, 1942.

Hartry Field. *Saving Truth from Paradox.* Oxford University Press, Oxford, 2008.

Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

Jean-Yves Girard. Light linear logic. *Information and Computation*, 143:175–204, 1998.

Laurence Goldstein. Epimenides and Curry. *Analysis*, 46:117–121, 1986.

Vjačeslav Grišin. A nonstandard logic and its applications to set theory. In *Studies in Formalized Languages and Nonclassical Logics*, pages 135–171. Nauka, Moscow, 1974.

Vjačeslav Grišin. Predicate and set-theoretic calculi based on logics without contractions. *Izvestija Akademij Nauk SSSR Serija Matematičeskaja*, 45:47–68, 1981.

Anil Gupta and Nuel Belnap. *The Revision Theory of Truth.* MIT Press, Cambridge MA, 1993.

Saul Kripke. Outline of a theory of truth. *The Journal of Philosophy*, 72:690–716, 1975.

Dan López de Sa and Elia Zardini. Does this sentence have no truthmaker? *Analysis*, 66: 154–157, 2006.

Dan López de Sa and Elia Zardini. Truthmakers, knowledge and paradox. *Analysis*, 67: 242–250, 2007.

Dan López de Sa and Elia Zardini. No-no. Paradox and consistency. *Analysis*, 71:472–478, 2011.

David Makinson. The paradox of the preface. *Analysis*, 25:205–207, 1965.

Vann McGee. *Truth, Vagueness, and Paradox.* Hackett, Indianapolis, 1991.

Chris Mortensen and Graham Priest. The truth teller paradox. *Logique et Analyse*, 24:381–388, 1981.

Gabriel Nuchelmans. A 17th-century debate on the *consequentia mirabilis*. *History and Philosophy of Logic*, 13:43–58, 1992.

Annibale Pastore. *La logica del potenziamento*. Rondinella, Naples, 1936.

Uwe Petersen. Logic without contraction as based on inclusion and unrestricted abstraction. *Studia Logica*, 64:365–403, 2000.

Uwe Petersen. *Diagonal Method and Dialectical Logic*. Der Andere, Osnabrück, 2002.

Graham Priest. *Beyond the Limits of Thought*. Oxford University Press, Oxford, 2nd edition, 2003.

Graham Priest. *In Contradiction*. Oxford University Press, Oxford, 2nd edition, 2006.

Roy Sorensen. *Vagueness and Contradiction*. Oxford University Press, Oxford, 2001.

Vladimir Stepanov. Propositional logics of reflexive sentences. *Naučno-Techničeskaja Informacija Serija 2*, 5:8–14, 2007.

Igor Urbas. Dual-intuitionistic logic. *Notre Dame Journal of Formal Logic*, 37:440–451, 1996.

Richard White. A demonstrably consistent type-free extension of the logic BCK. *Mathematica Japonica*, 32:149–169, 1987.

Richard White. A consistent theory of attributes in a logic without contraction. *Studia Logica*, 52:113–142, 1993.

Elia Zardini. Truth and what is said. *Philosophical Perspectives*, 22:545–574, 2008.

Elia Zardini. Truth without contra(di)ction. *The Review of Symbolic Logic*, 4:498–535, 2011.

Elia Zardini. Truth preservation in context and in its place. In Catarina Dutilh-Novaes and Ole Hjortland, editors, *Insolubles and Consequences*, pages 249–271. College Publications, London, 2012.

Elia Zardini. Naive *modus ponens*. *Journal of Philosophical Logic*, 2013a. Forthcoming.

Elia Zardini. Naive truth and naive logical properties. *The Review of Symbolic Logic*, 2013b. Forthcoming.

Elia Zardini. It is not the case that [P and 'It is not the case that P' is true]. ms, 2013c.

Elia Zardini. Naive restricted quantification. ms, 2013d.

Elia Zardini. The opacity of truth. ms, 2013e.