Higher-Order Sorites Paradox*

Elia Zardini

Instituto de Investigaciones Filosóficas

Universidad Nacional Autónoma de México

Northern Institute of Philosophy

University of Aberdeen

elia.zardini@abdn.ac.uk

July 27, 2011

Abstract

The naive theory of vagueness holds that the vagueness of an expression consists in its failure to draw a sharp boundary between positive and negative cases. The naive theory is contrasted with the nowadays

*This paper has long been in the works (an early reference to it can be found already in Greenough [2005], p. 180, fn 10). Earlier versions of the paper's material have been presented in 2004 at the Arché Vagueness Seminar; in 2005 at the 9th NPAPC (University of York), where David Efird gave a valuable response; in 2006 at a Vagueness Workshop (University of Freibourg, in Ovronnaz), where Reto Givel gave another valuable response, and at the 6th GAP Conference on *Philosophy: Foundations and Applications* (Berlin Free University); in 2007 at the 1st UCLA/USC Graduate Student Conference in Philosophy (UCLA), where Gabe Rabin gave yet another valuable response. I would like to thank all these audiences for very stimulating comments and discussions. Special thanks go to Dan López de Sa, Hartry Field, Mario Gómez-Torrente, Patrick Greenough, Richard Heck, Sebastiano Moruzzi, Peter Pagin, Walter Pedriali, Steve Read, Robbie Williams, Tim Williamson, Crispin Wright and to several anonymous referees. In writing this paper, I have benefitted, at different stages, from an AHRC Doctoral Award, a RIP Jacobsen Fellowship, an AHRC Postdoctoral Research Fellowship and a UNAM Postdoctoral Research Fellowship, as well as from partial funds from the project FFI2008-06153 of the Spanish Ministry of Science and Innovation on *Vagueness, Physics, Metaphysics and Metametaphysics*, from the project CONSOLIDER-INGENIO 2010 CSD2009-00056 of the Spanish Ministry of Science and Innovation on *Philosophy of Perspectival Thoughts and Facts* (PERSP) and from the European Commission's 7th Framework Programme FP7/2007-2013 under grant FP7-238128 for the European Philosophy Network on *Perspectival Thoughts and Facts* (PETAF).

dominant approach to vagueness, holding that the vagueness of an expression consists in its presenting borderline cases of application. The two approaches are briefly compared in their respective explanations of a paramount phenomenon of vagueness: our ignorance of any sharp boundary between positive and negative cases. These explanations clearly do not provide any ground for choosing the dominant approach against the naive theory. The decisive advantage of the former over the latter is rather supposed to consist in its immunity to any form of sorites paradox. But another paramount phenomenon of vagueness is higher-order vagueness: the expressions (such as 'borderline' and 'definitely') introduced in order to express in the object language the vagueness of the object language are themselves vague. Two highly plausible claims about higher-order vagueness are articulated and defended: the existence of "definitely$^{\omega}$" positive and negative cases and the "radical" character of higher-order vagueness itself. Using very weak logical principles concerning vague expressions and the 'definitely'-operator, it is then shown that, in the presence of higher-order vagueness as just described, the dominant approach is subject to higher-order sorites paradoxes analogous to the original ones besetting the naive theory, and therefore that, against the *communis opinio*, it does not fare substantially better with respect to immunity to any form of sorites paradox.

Keywords: borderline cases, higher-order vagueness, ignorance, sorites paradox, tolerance.

1 Introduction and Overview

The *naive theory of vagueness* holds that, roughly and at least in the basic cases, the vagueness of an expression consists in its *tolerance*, namely its *failure to draw a sharp boundary between positive and negative cases*, where a "*sharp boundary*" is understood to be a positive case immediately followed in the relevant ordering by a negative case (I argue in favour of the naive theory in Zardini [2011b]). The theory falls within what may well be called 'the *traditional approach to vagueness*',[1] which, among the various phenomena of vagueness, primarily focusses on that consisting in the fact that a vague expression at least *seems* to fail to draw a sharp

[1] Many different kinds of quotations will be employed in this paper: for example, simple quotation, quasi-quotation, autonymous quotation and combinations thereof. As usual, simple quotes and display will ambiguously be used to signal any of these kinds of quotation. As usual, context will disambiguate.

boundary between positive and negative cases.[2,3] The traditional approach—and, *a fortiori*, the naive theory—is nowadays mostly rejected in favour of what may well be called 'the *dominant approach to vagueness*', which, among the various phenomena of vagueness, primarily focusses instead on that consisting in the fact that a vague expression *presents borderline cases*.[4] Of course, either approach typically also tries to address the phenomenon that the other approach primarily focusses on (and other phenomena as well), but both approaches do so by building and expanding on what they say about the phenomenon they do primarily focus on.

Appealing as it may appear at a first glance, the naive theory of vagueness has on reflection been rejected by almost every author on the grounds that its characteristic claim of tolerance makes it subject to standard *sorites paradoxes*. Opponents of the naive theory have thus sought to find a suitably weaker claim which, while no longer affected by sorites paradoxes, still manages to capture (a great deal of) what the naive theorist tries to capture with her claim of tolerance. Within the dominant approach, the natural fall-back has been a claim asserting the existence of borderline cases. That such a retreat is really safe has been assumed without much argument, save for briefly noting that there does not seem to be any easy way of generating sorites paradoxes out of the materials afforded by claims that merely assert the existence of borderline cases. The alleged advantage of the dominant approach over the naive theory has almost wholly relied on this article of faith. It is high time to shake it.

The rest of the paper is organized as follows. Section 2 recalls a surprising phenomenon of ignorance related to vagueness and sketches what explanation is given thereof by the naive theory and by the dominant approach respectively,

[2]The label 'traditional' is justified by the fact that the way in which, starting from the ancient Greeks, vagueness has typically and for long occupied Western philosophy has primarily been as a problem of apparent lack of sharp boundaries (see Williamson [1994], pp. 8–35 for some details and references).

[3]Being naive, the naive theory also holds that there are both positive cases and negative cases, and that these are positive-only (i.e. not also negative) and negative-only (i.e. not also positive) respectively. Thus, the naive theory substantially diverges from—and is much more in keeping with common sense than—some other theories falling within the traditional approach which also endorse full-fledged tolerance principles, such as e.g. the *nihilist* theory advocated by Unger [1979] (according to which everything is a negative case). In this respect, the naive theory is much closer to another, more fashionable kind of theory falling within the traditional approach, i.e. *contextualism* (see e.g. Raffman [1994]; Fara [2000]). The naive theory however substantially diverges also from standard contextualist theories, exactly in its endorsement of full-fledged tolerance principles (although standard contextualist theories spend much effort in explaining why such principles seem to be true, they ultimately deem them untrue). Some reasons for preferring the naive theory over standard contextualist theories are given in Sweeney and Zardini [2011].

[4]The label 'dominant' is justified by the fact that typical semantic, ontic, epistemic, psychological and conceptual-role theories are primarily theories about borderline cases (see section 2.1 for some details and references).

demonstrating how the latter is committed to principles in certain respects similar to tolerance principles. Section 3 rehearses another phenomenon of vagueness, higher-order vagueness, articulating and defending two main claims about it. Section 4 uses these claims to develop a higher-order sorites paradox, showing that the dominant approach is after all no less paradoxical than the naive theory. Section 5 draws the conclusions which follow from the higher-order sorites paradox for the dialectic between the naive theory and the dominant approach.

2 Ignorance

2.1 Ignorance and Borderlineness

Consider the series $\mathbb{S}$ of natural numbers from 0 to $1,000,000$, the vague predicate 'A person with ξ hairs on her scalp is bald' (henceforth '$B\xi$') and a conversational context (a fairly common one, I suppose) where 0 and $1,000,000$ are, respectively, indisputable positive and negative cases for the application of 'B'. If there is a sharp boundary in $\mathbb{S}$ between the Bs and the $\neg B$s,[5] where does it lie? It is yet another phenomenon of vagueness that we simply seem to be unable to provide a knowledgeable identification of such a boundary. Properly articulating this phenomenon requires some care: what is phenomenologically manifest is simply that *we participants of the vagueness debate* are unable to provide such an identification *on the basis of the generally available methods used to track* B*ness and* $\neg B$*ness.* But maybe someone *else* is able to use these very *same* methods so as to provide such an identification; or maybe there are some *other* methods which would enable *us* to provide such an identification. In the following, no more than the above cautious articulation will be assumed (and apparently unqualified use of 'know' and its like should be understood accordingly).

It is generally agreed that our surprising inability to provide a knowledgeable identification of the sharp boundary between the Bs and the $\neg B$s is due to the vagueness of 'B'.[6] But how exactly is the latter supposed to explain the former? For the naive theory, the explanation is most straightforward: vagueness implies lack of sharp boundaries, whereas, by factivity of knowledge, knowledge of a sharp boundary would imply the existence of a sharp boundary. The story is a bit more complicated for the dominant approach.[7] The approach has it that the vagueness

[5] To preserve clarity, I will often help myself to a moderate, hopefully self-explanatory regimentation of my language into "Loglish".

[6] Arguably, this phenomenon of ignorance extends from lack of *knowledge* to a more general lack of *warrant*, but, in order to avoid complexities which seem irrelevant for the purposes of this paper, we will focus only on this peculiar lack of knowledge and take it as our guide to vagueness (see in particular section 3.2).

[7] From here until the end of this section, I will expose a very abstract and minimal explanatory structure, some of whose assumptions will be differently understood and justified by different theories falling within that approach (I give an important example of

of 'B' explains our epistemic inability insofar as 'B' presents borderline cases of application in $\mathbb{S}$: objects which are neither definitely B nor definitely $\neg B$ (I will have something more to say on the relation between the property of being borderline and the property of being definite in section 3.1). Letting henceforth our objectual variables range over the elements of $\mathbb{S}$ (unless otherwise specified) and '$\mathcal{D}$' be a 'definitely'-operator, the defender of the dominant approach thus accepts the *borderlineness* principle:

(B) $\exists x(\neg\mathcal{D}Bx \wedge \neg\mathcal{D}\neg Bx)$.

Before continuing the exposition of the explanation of the phenomenon of ignorance given by the dominant approach, a clarificatory comment on my assumptions concerning the 'definitely'-operator is in order. It is of course a major task of any theory of vagueness availing itself to this operator to provide a theoretical explication of what the property of being definite consists in. The contemporary literature on vagueness does in effect provide a wide range of (first-pass) proposals both for an *explicit* definition of 'Definitely, φ':

Semantic: Speakers' practices determine a sufficient truth condition for 'φ', and the condition is satisfied (see e. g. Fine [1975]; McGee and McLaughlin [1995]);

Ontic: It is a fact of the matter that φ (see e.g. Tye [1990]);

Epistemic: No obstacle of a certain kind prevents knowledge that φ (see e.g. Sorensen [1988], pp. 199–252; Williamson [1994]);

Psychological: It is correct to hold a standard (possibly partial) belief to the effect that φ (see e.g. Schiffer [2003], pp. 178–237)

and for an *implicit* definition of it by means of a specification of the conceptual role of the 'definitely'-operator (see e.g. Field [1994], pp. 409–422; Field [2003]). In this respect, let me stress right at the outset that all the arguments to follow will be fairly neutral with respect to the specific interpretation of the 'definitely'-operator, turning only on basic common features required by its use in describing borderline cases and on a minimal **KT**-like normal modal logic validating its *factivity* and *closure under finite logical consequence*:

(F) $\vdash \mathcal{D}\varphi \supset \varphi$;[8]

this in fn 10). While it is beyond the scope of this paper to justify the (highly plausible) sociological claim that that is in effect the structure implicitly or explicitly presupposed by the theories in question, it clearly constitutes the most natural and appealing explanatory strategy available within the dominant approach.

[8]Throughout, I will use '$\vdash$' to denote the contextually relevant consequence relation, always understood as consisting in a logic for a standard first-order language plus a logic for '$\mathcal{D}$'. I will also take '$\vdash$' to create a quotation environment for formulae occurring on its sides.

(C) If $\varphi_0, \varphi_1, \varphi_2 \ldots \varphi_i \vdash \psi$, then $\mathcal{D}\varphi_0, \mathcal{D}\varphi_1, \mathcal{D}\varphi_2 \ldots \mathcal{D}\varphi_i \vdash \mathcal{D}\psi$.[9]

To come back then to our main thread, how exactly is (B) supposed to explain our surprising epistemic inability? Two auxiliary claims are—to my knowledge—pretty much universally and—to my mind—very plausibly accepted. The first concerns the *borderline location of sharp boundaries*:

(L) $\exists x(\neg\mathcal{D}Bx \wedge \neg\mathcal{D}\neg Bx) \supset \forall x((Bx \wedge \neg Bx') \supset (\neg\mathcal{D}Bx \wedge \neg\mathcal{D}\neg Bx \wedge \neg\mathcal{D}Bx' \wedge \neg\mathcal{D}\neg Bx'))$,

where '$'$' is the standard successor functor, which is well-defined on $\mathbb{S}$. The second concerns *lack of asymmetric knowledge of adjacent borderline cases*:

(K) If two adjacent objects are borderline cases, it is not known that one is B and the other $\neg B$.[10]

Factivity of knowledge reduces then to inconsistency the set consisting in (B), (L), (K) and the further claim that it is known where the sharp boundary between the Bs and the $\neg B$s lies. Hence, keeping fixed (B), (L) and (K), an explanation of the phenomenon of ignorance revolving around (B) nicely follows for the dominant approach.[11] So far so good. However, we now turn to other more problematic commitments of that approach.

[9]Principles along the lines of (F) and (C) have both come into question in another area where the notion of being definite is supposed by many to do some helpful work—namely, in the area of the semantic paradoxes. On pain of revenge, classical theories of truth such as the one developed in McGee [1991] reject what is in effect (F) and non-classical theories of truth such as the one developed in Field [2008] must reject the conditional version of the countable strengthening of (C). No matter what one thinks of these moves in the context of the semantic paradoxes, I don't think that a kindred rejection of either (F) or (C) can be motivated in the case of vagueness. Thanks to an anonymous referee for discussion of (C) in particular.

[10]A stronger, almost universally accepted ignorance principle—very much suggested by many of the theoretical explications of the property of being definite we have just reviewed—concerns *lack of knowledge of borderline cases*:

(SK) If an object is a borderline case, it is neither known to be B nor known to be $\neg B$.

(SK) has however been forcefully rejected by some theorists of vagueness (see e.g. Wright [2001]). Nevertheless, I believe that at least some of these theorists would still accept the weaker ignorance principle (K) stated in the text, which is all is required for the current explanation of the phenomenon of ignorance to go through (see also fn 11).

[11]It may at least be broadly in the spirit of the type of view defended by Dorr [2003]; Barnett [2011] to reject the characterization of the phenomenon in terms of lack of knowledge and opt instead for a weaker characterization of it in terms of lack of *definite* knowledge. On this option, the explanation given in the text would still retain its compellingness (as an explanation of the phenomenon characterized in this weaker way) by replacing in (K) lack of knowledge with lack of definite knowledge. Thanks to an anonymous referee for raising this issue.

2.2 Indefiniteness and Tolerance

To start with, reflect that $\mathbb{S}$ is such that the *trichotomy* principle:

(T) $\forall x \forall y (x < y \vee y < x \vee x = y)$

obviously holds. Moreover, the meanings of 'B' and '$\mathcal{D}$' have been sufficiently explicated so as to validate the *monotonicity* principle:

(M) $\forall x ((\mathcal{D}Bx' \supset \mathcal{D}Bx) \wedge (\mathcal{D}\neg Bx \supset \mathcal{D}\neg Bx'))$

(for simplicity's sake, I am acquiescing in the usual, harmlessly false assumption that baldness is just a matter of the number of hairs on one's scalp). We can then establish:

Theorem 1. *Together with* (T) *and* (M)*,* (B) *entails the* indefiniteness *principle:*

(I) $\neg \exists x (\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$*.*

Proof. Suppose for *reductio* that there is a definite sharp boundary between the Bs and the $\neg B$s (that is, that there is an object x such that $\mathcal{D}Bx \wedge \mathcal{D}\neg Bx'$). Let an arbitrarily chosen object a be such. Consider then an arbitrarily chosen object b. By (T), b is either smaller than a or greater than a or identical with a. If b is smaller than a, then, by (M), b is $\mathcal{D}B$, and therefore $\mathcal{D}B \vee \mathcal{D}\neg B$. If b is greater than a, then, by (M), b is $\mathcal{D}\neg B$, and therefore $\mathcal{D}B \vee \mathcal{D}\neg B$. If b is identical with a, then, by Leibniz's Law, b is $\mathcal{D}B$, and therefore $\mathcal{D}B \vee \mathcal{D}\neg B$. (Note that intersubstitutability of identicals in '$\mathcal{D}$'-contexts may very well result problematic, at least under certain interpretations of '$\mathcal{D}$'. This delicate issue lies however outside the scope of this paper.) Therefore, reasoning by cases, b is $\mathcal{D}B \vee \mathcal{D}\neg B$. Therefore, by an uncontroversial De Morgan law, b is $\neg(\neg\mathcal{D}B \wedge \neg\mathcal{D}\neg B)$. Therefore, by universal generalization, everything is $\neg(\neg\mathcal{D}B \wedge \neg\mathcal{D}\neg B)$. Therefore, by an uncontroversial quantified De Morgan law, nothing is $\neg\mathcal{D}B \wedge \neg\mathcal{D}\neg B$. Therefore, by existential instantiation (discharging the supposition that $\mathcal{D}Ba \wedge \mathcal{D}\neg Ba'$), nothing is $\neg\mathcal{D}B \wedge \neg\mathcal{D}\neg B$. Contradiction with (B). Therefore, by *reductio*, there is no definite sharp boundary between the Bs and the $\neg B$s (cf Greenough [2003], pp. 270–271).

□

(I) is certainly not as straightforwardly paradoxical as the stronger, naive *tolerance* principle:

(TOL) $\neg \exists x (Bx \wedge \neg Bx')$.

On the one hand, (TOL) says that at no point in $\mathbb{S}$ does a single step (that is, a step from a number to its successor) bring us from the Bs to the $\neg B$s, which may be thought to capture well what the apparent *smoothness* of $\mathbb{S}$ with respect to Bness consists in, but—by the standard *sorites paradox*—seems to be inconsistent with 0's being B and $1,000,000$'s being $\neg B$. More in detail, from (TOL) we have that it is not the case that [999,999 is B and 1,000,000 is $\neg B$],[12] which, together with $1,000,000$'s being $\neg B$, yields that 999,999 is $\neg B$. Yet, from (TOL) we also have that it is not the case that [999,998 is B and 999,999 is $\neg B$], which, together with the previous lemma that 999,999 is $\neg B$, yields that that 999,998 is $\neg B$. 999,998 structurally identical arguments eventually lead to the conclusion that 0 is $\neg B$, thereby contradicting 0's being B.

On the other hand, (I) only says that at no point in $\mathbb{S}$ does a single step bring us from the $\mathcal{D}B$s to the $\mathcal{D}\neg B$s, which may be thought already to suffice for capturing what the apparent smoothness of $\mathbb{S}$ with respect to Bness consists in, and, crucially, seems to be consistent with 0's being B and $1,000,000$'s being $\neg B$. However, some highly plausible claims concerning the phenomenon of higher-order vagueness suffice to show that principles of the form of (I) (and, therefore, principles of the form of (B)) are, in a subtler way, just as paradoxical as (TOL)—which, as remarked in section 2.1, was the most obvious candidate for a characterization of the nature of vagueness from which an explanation of our epistemic inability would nicely follow.

3 Higher-Order Vagueness

3.1 Being Definite$^{\omega}$

In accounting for the effects in $\mathbb{S}$ of the vagueness of 'B', the new compound predicate '$\mathcal{D}B$' has been used by the dominant approach. Arguably, this new predicate is itself vague: for one, we seem to be just as much unable to provide a knowledgeable identification of the sharp boundary between the $\mathcal{D}B$s and the $\neg\mathcal{D}B$s as we are concerning the sharp boundary between the Bs and the $\neg B$s. The operator '$\mathcal{D}$' was introduced in order to express in an enriched object language the vagueness of the original object language (in particular, the vagueness of 'B'). That this operator itself gives rise to new vague expressions crucially containing it is yet another typical phenomenon of vagueness, which has traditionally been called '*higher-order vagueness*'.

Two claims concerning higher-order vagueness seem to be highly plausible. After stating each, I will defend its plausibility with some suggestive considerations, but I should like to stress that the ultimate commitment to these claims will of course partly depend on a specific theory of what the property of being definite consists in—an issue about which I am deliberately remaining neutral (see section 2.1). The main point of this paper being rather to show that very untoward consequences follow for the dominant approach from these two claims, it will suffice to bring out their high and largely theory-neutral plausibility, thereby articulating

[12] Throughout, I use square brackets to disambiguate constituent structure in English.

some of the costs and challenges awaiting a theory that should ultimately reject either of them.

Before stating and defending these claims, we need to introduce some more notation. For every natural number i, let '$\mathcal{D}^i$' be a shorthand for the expression obtained by concatenating the empty string with i occurrences of '$\mathcal{D}$'. Assuming the availability in the object language of the expressive resources afforded by substitutional quantification, we can now introduce a '*definitely*$^\omega$'-operator '$\mathcal{D}^\omega$' by letting '$\mathcal{D}^\omega\varphi$' be satisfied iff '$\Pi i \mathcal{D}^i\varphi$' is.

This is a particularly neutral way of introducing transfinite orders of the property of being definite without presupposing much with respect to the semantics of '$\mathcal{D}$'. On some semantics for a vague language (for example, supervaluationism and epistemicism), '$\mathcal{D}$' usually receives an interpretation in terms of possible-world semantics, being in effect treated as a necessity-like operator. In such a semantic framework, '$\mathcal{D}^\omega$' can be interpreted as the *ancestral* of '$\mathcal{D}$', thereby meaning that the possible-world semantics for '$\mathcal{D}^\omega$' uses as accessibility relation the ancestral (that is, the transitive closure) of the accessibility relation used by the possible-world semantics for '$\mathcal{D}$' (see Williamson [1994], p. 160). On other semantics for a vague language (for example, many-valued), '$\mathcal{D}$' usually receives an interpretation in terms of algebraic semantics, being in effect treated as a lowering operation on the structure of values. In such a semantic framework, '$\mathcal{D}^\omega$' can be interpreted as the *greatest lower bound* of '$\mathcal{D}$', thereby meaning that the algebraic semantics for '$\mathcal{D}^\omega$' assigns to it the operation that takes a value v to the greatest lower bound of the set of values to which v is taken by the operations denoted by finite concatenations of '$\mathcal{D}$' (see Field [2007], p. 114).

The first claim concerning higher-order vagueness is then that 0 (by assumption, an indisputable positive case of Bness) *is* $\mathcal{D}^\omega B$ and that $1,000,000$ (by assumption, an indisputable negative case of Bness) *is* $\mathcal{D}^\omega\neg B$:

(O) $\mathcal{D}^\omega B0 \wedge \mathcal{D}^\omega\neg B1,000,000$.

(O) is validated by some of the most influential semantics for a vague language including a 'definitely'-operator: supervaluationist semantics assigning super-truth to '$\mathcal{D}\varphi$' if super-truth is assigned to φ and allowing for '$B0$' and '$\neg B1,000,000$' to be super-true (see e.g. Fine [1975]), many-valued semantics assigning full truth to '$\mathcal{D}\varphi$' if full truth is assigned to φ and allowing for '$B0$' and '$\neg B1,000,000$' to be fully true (see e.g. Sanford [1975])—as well as, I suspect, being taken for granted in most of the literature (notable exceptions are Williamson [1994], pp. 229, 232–233; Williamson [2002], pp. 145–146; Dorr [2010]).[13]

[13] Although this would deserve a separate treatment, I think that these authors' opposition to (O) crucially depends on relying on a basic possible-world semantics and on an understanding of the accessibility relation that are ill-suited for modelling important features of iterations of '$\mathcal{D}$', some of which are actually brought out by the considerations in favour of (O) to follow in the text (**?** contains a detailed discussion of similar issues arising for knowledge rather than definiteness, plus an alternative semantic proposal).

More importantly, (O) is very intuitive independently of the specific interpretation received by the 'definitely'-operator (see section 2.1), for it amounts to saying that an indisputable positive or negative case does *not* exhibit vagueness at *any* order. For example, it is very intuitive that it is not vague that 0 is B. And it is very intuitive that it is in turn not vague that there is no such vagueness. And it is very intuitive that it is in turn not vague that there is no such vagueness. *Et sic in infinitum.*

Even more importantly, (an analogue of) (O) seems unavoidable for a vague predicate like 'ξ is close to 0 and [ξ is far from 1,000,000 or identical with 0]' as applied to 0 (positively) and 1,000,000 (negatively). This is so because it seems unavoidable that facts about identity either definitely$^{\omega}$ hold or definitely$^{\omega}$ fail to hold, that x-is-close-to-y is definitely$^{\omega}$ reflexive and that x-is-far-from-y is definitely$^{\omega}$ irreflexive—and, by (C), these claims will jointly entail (the relevant analogue of) (O).[14] Notice that the principles concerning the two vague relations x-is-close-to-y and x-is-far-from-y are crucially different from sentences like '$B0$' and '$\neg B1,000,000$', and hence that the assumption of the definite$^{\omega}$ truth of the former does not beg the question against someone who doubts the definite$^{\omega}$ truth of the latter.[15] For example, '$B0$', although indisputably true, is a predication to a number of a property that other numbers (indisputably) lack, and I take it that it is this circumstance that is largely responsible for the opposition to (O) (notably, for the opposition stemming from the assumption that each new iteration of '$\mathcal{D}$' introduces an additional margin from the $\neg B$s). The situation is crucially different for '0 is close to itself': that sentence is not only indisputably true, but is also a predication to a number of a property that also every other number (indisputably) has. Given this indisputable *universality*, it is hard to see how the sentence could fail to be definitely$^{\omega}$ true.

Finally, (O) can be justified by various considerations concerning the *function* which the 'definitely'-operator is supposed to serve and which allegedly constitutes our primary grasp of it. The issues here are surprisingly deep and difficult, deserving a far more extended treatment than I can afford in this paper, but let me sketch just one such consideration. On the dominant approach, borderline cases are typically supposed to be not just a theoretical construction needed, for every i, in the explanation of our epistemic inability concerning the sharp boundary between the $\mathcal{D}^iB$s and the $\neg\mathcal{D}^iB$s; rather, they are typically supposed to be *manifested* in our experience of being confronted with "hard cases" of $\mathcal{D}^iB$ness—that is, cases where, even after taking in all the relevant '$\mathcal{D}^iB$'-free information, we competent speakers for '$\mathcal{D}^iB$' feel unconfident both in unqualifiedly applying '$\mathcal{D}^iB$' to the case in question and in unqualifiedly applying '$\neg\mathcal{D}^iB$' to it.[16]

[14] Strictly speaking, what is needed is closure of the property of being definite$^{\omega}$ under finite logical consequence. It is easy however to see that that is entailed by closure of the property of being definite under finite logical consequence, i.e. by (C).

[15] Thanks to an anonymous referee for raising this worry.

[16] Indeed, from within the dominant approach, it may be argued that, in general, our

So, go along $\mathbb{S}$ starting from 0 and considering, case by case, the application of 'B'. You are extremely confident in applying 'B' to 0, but, sooner or later, you will find yourself confronted with cases where, even after taking in all the relevant 'B'-free information, you competent speaker for 'B' feel unconfident both in unqualifiedly applying 'B' to the case in question and in unqualifiedly applying '$\neg B$' to it: these are the borderline cases of Bness—that is, cases which are neither $\mathcal{D}B$ nor $\mathcal{D}\neg B$. Your situation with respect to 0 was most clearly *not like that* (most clearly, you were on the contrary extremely confident in applying 'B' to it): reflecting on the *change* occurred between 0 and the borderline cases of Bness, you infer that 0 is *not* a borderline case of Bness—that is, that it is $\neg(\neg\mathcal{D}B \wedge \neg\mathcal{D}\neg B)$. And exactly because you are extremely confident that 0 is B, by (F) and contraposition you also infer that it is $\neg\mathcal{D}\neg B$. Therefore, by adjunction and *reductio*, you infer that 0 is $\neg\neg\mathcal{D}B$. Therefore, by double-negation elimination, you infer that it is $\mathcal{D}B$.

Before continuing with the present consideration in favour of (O), a couple of logical comments might be useful for the interested reader (those readers that are not so interested can jump to the next paragraph). Firstly, the previous reasoning is of course *intuitionistically* invalid at its very last step. It is possible to adapt the argument from here on in order to make do without that step. I leave to the interested reader the details of this as well as those of similar modifications that would be needed at later stages of the argument. However, note that, adopting instead the less usual definition of definite cases in terms of borderline cases suggested in fn 16, a legitimate worry concerning double-negation elimination is no longer possible, since, now, for something to be definitely F it takes no more than for it to be F and not to be borderline F (see Wright [2001] for an intuitionist approach to the logic of vagueness).[17] Secondly, the application of *reductio* at the second last step of the previous reasoning is *paraconsistently* (and hence relevantly) invalid.

---

grasp of the 'definitely'-operator flows from our grasp of a more basic 'borderline'-operator, so that definite cases should be defined in terms of borderline cases by saying that an object is definitely F iff it is F and not borderline F (rather than, as more usual, defining borderline cases in terms of definite cases by saying that an object is borderline F iff it is neither definitely F nor definitely $\neg F$). Throughout this paper, I am mostly sticking to the more usual definition, save for remarking in a couple of places to follow in the text how adoption of the alternative, less usual definition would simply make my argument easier.

[17]In general, no legitimate worry concerning double-negation elimination with respect to 'F' over a particular range of application is possible if 'F' is known to be *exclusive and exhaustive* with respect to 'G' over that range (in the sense that, if something in the range is G, it is $\neg F$, and that, if something in the range is $\neg G$, it is F). By exclusivity, if something is G, it is $\neg F$, wherefore, by contraposition, if something is $\neg\neg F$, it is $\neg G$. By exhaustivity, if something is $\neg G$, it is F. By transitivity of the conditional, if something is $\neg\neg F$, it is F. (Let 'F' be 'definitely B' and 'G' be 'borderline B': the less usual definition of definite cases in terms of borderline cases suggested in fn 16 makes 'definitely B' (exclusive and) exhaustive with respect to 'borderline B' over the Bs, which is the range of interest since we can safely assume that the object of our predications, 0, is B.) See Zardini [2011c] for another application of this point in the case of vagueness.

Additional, plausible assumptions concerning intensional connections between 0's being $\neg\mathcal{D}\neg B$ and 0's being $\neg\neg\mathcal{D}B$ (licensing the inference from the former to the latter and spelling out in effect the established fact that 0 is not a borderline case) would have to be made in order to turn the argument into a relevantly (and hence paraconsistently) valid one. Again, I leave to the interested reader the details of this as well as those of similar modifications that would be needed at later stages of the argument. Alternatively, the same strategy of adopting the less usual definition of definite cases in terms of borderline cases suggested in fn 16 would do in this case as well (see Hyde [1997] for a paraconsistent approach to the logic of vagueness).[18]

To continue then with the present consideration in favour of (O), we need to add a more specific but plausible claim about the behaviour of your confidence during the previous process. The claim is that you were *maximally* confident in asserting, on reflection, the fundamental difference between your situation with respect to 0 and your situation with respect to the borderline cases of Bness. This claim is justified by the observation that the fact that 0 is not a hard case of Bness is as clear as facts about negative cases of the property of being a hard case can get (simply given the sheer amount of your confidence in applying 'B' to 0), and that there is no reason to think that this kind of fact cannot get as clear as to warrant maximal confidence (at least assuming—as I will do—that ordinary kinds of facts involving a vague property can get that clear, for facts about negative cases of the property of being a hard case would then seem to be just another ordinary kind of fact involving a vague property). But, together with the assumption that 0 is B, that there is such a fundamental difference between your situation with respect to 0 and your situation with respect to the borderline cases of Bness is the only other substantial assumption on which the previous conclusion that 0 is $\mathcal{D}B$ depends. Hence, assuming that your confidence is governed by standard probabilistic constraints, your confidence in applying '$\mathcal{D}B$' to 0 should be at least as high as your confidence in applying 'B' to it—indeed, given that you are also maximally confident in asserting (F), your confidence in applying '$\mathcal{D}B$' to 0 should equal your confidence in applying 'B' to it. Since the latter is extremely high,[19] so should be the former.

So, go along $\mathbb{S}$ starting again from 0 but, this time, considering, case by case, the application of '$\mathcal{D}B$'. We have just established that you are extremely confident in applying '$\mathcal{D}B$' to 0, but, sooner or later, you will find yourself confronted with cases where, even after taking in all the relevant '$\mathcal{D}B$'-free information, you

[18]Thanks to Steve Read for discussion of the bearing of paraconsistency and relevance on this step.

[19]'Maximally high' I would say, on the strength of a consideration analogous to that just used to establish that you were maximally confident in asserting, on reflection, the fundamental difference between your situation with respect to 0 and your situation with respect to the borderline cases of Bness. Notice however that this second maximality claim is not required by the present consideration in favour of (O).

competent speaker for '$\mathcal{D}B$' feel unconfident both in unqualifiedly applying '$\mathcal{D}B$' to the case in question and in unqualifiedly applying '$\neg\mathcal{D}B$' to it: these are the borderline cases of $\mathcal{D}B$ness—that is, cases which are neither $\mathcal{D}\mathcal{D}B$ nor $\mathcal{D}\neg\mathcal{D}B$. Having followed through the previous reasoning, your situation with respect to 0 was most clearly *not like that* (most clearly, you were on the contrary extremely confident in applying '$\mathcal{D}B$' to it): reflecting on the *change* occurred between 0 and the borderline cases of $\mathcal{D}B$ness, you infer that 0 is *not* a borderline case of $\mathcal{D}B$ness—that is, that it is $\neg(\neg\mathcal{D}\mathcal{D}B \wedge \neg\mathcal{D}\neg\mathcal{D}B)$. And exactly because we have just established that you are extremely confident that 0 is $\mathcal{D}B$, by (F) and contraposition you also infer that it is $\neg\mathcal{D}\neg\mathcal{D}B$. Therefore, by adjunction and *reductio*, you infer that 0 is $\neg\neg\mathcal{D}\mathcal{D}B$. Therefore, by double-negation elimination, you infer that it is $\mathcal{D}\mathcal{D}B$. (Logical comments analogous to those made earlier apply of course to these inferences.)

Again, I claim that you were plausibly *maximally* confident in asserting, on reflection, the fundamental difference between your situation with respect to 0 and your situation with respect to the borderline cases of $\mathcal{D}B$ness. This claim is justified by the observation that the fact that 0 is not a hard case of $\mathcal{D}B$ness is as clear as facts about negative cases of the property of being a hard case can get (simply given the sheer amount of your confidence in applying '$\mathcal{D}B$' to 0), and that there is no reason to think that this kind of fact cannot get as clear as to warrant maximal confidence. But together with the previous lemma that 0 is $\mathcal{D}B$, that there is such a fundamental difference between your situation with respect to 0 and your situation with respect to the borderline cases of $\mathcal{D}B$ness is the only other substantial assumption on which the previous conclusion that 0 is $\mathcal{D}\mathcal{D}B$ depends. Hence, assuming that your confidence is governed by standard probabilistic constraints, your confidence in applying '$\mathcal{D}\mathcal{D}B$' to 0 should be at least as high as your confidence in applying '$\mathcal{D}B$' to it—indeed, given that you are also maximally confident in asserting (F), your confidence in applying '$\mathcal{D}\mathcal{D}B$' to 0 should equal your confidence in applying '$\mathcal{D}B$' to it. Since, by the previous subargument, the latter is extremely high, so should be the former. *Et sic in infinitum.*

This "back-and-forth" argument establishes that, for every i, one should be equally confident in applying '$\mathcal{D}^iB$' to 0 as one is confident in applying 'B' to it, and so that one should be extremely confident in applying '$\mathcal{D}^iB$' to 0. Now, by standard probabilistic constraints, one's confidence in a *finite* conjunction each of whose conjunct entails the previous one should equal the greatest lower bound of one's confidence in each conjunct (which, by the finitude of the conjuncts, boils down to equalling one's confidence in the strongest conjunct). Although not forced by standard probabilistic constraints, it is plausible to extend this principle to *denumerable* conjunctions, and hold that one's confidence in a denumerable conjunction each of whose conjunct entails the previous one should equal the greatest lower bound of one's confidence in each conjunct. But since, by the back-and-forth argument, for every i, one should be extremely confident in applying '$\mathcal{D}^iB$' to 0, this extended principle entails that one should be extremely confident in applying '$\mathcal{D}^\omega B$' to it. Parallel arguments would of course establish the corresponding claim for '$\mathcal{D}^\omega \neg B$' and 1,000,000. Together, these arguments commit one to having a very high confidence in (O).[20]

3.2 Radical Higher-Order Vagueness

Save for the consideration concerning the unusual vague predicate 'ξ is close to 0 and [ξ is far from 1,000,000 or identical with 0]', all of the previous arguments in favour of (O) actually support the more general conclusion that, if it is indisputably the case that P, it is definitely$^\omega$ the case that P. I will henceforth assume their cogency, moving freely from its being indisputably the case that P to its being definitely$^\omega$ the case that P (and, when seems warranted, I will also move implicitly from its being the case that P to its being indisputably the case that P).

The second claim concerning higher-order vagueness is then that, for every i, *'$\mathcal{D}^i B$' is definitely$^\omega$ vague*. We can establish this claim by reflecting that, for every i, we seem to be just as much unable to provide a knowledgeable identification of the sharp boundary between the $\mathcal{D}^i B$s and the $\neg\mathcal{D}^i B$s as we are concerning the sharp boundary between the Bs and the $\neg B$s (and definitely$^\omega$ so, by the assumptions of the previous paragraph), and that there seems to be no relevant difference as regards the *source* of these various epistemic inabilities (and definitely$^\omega$ so, by the assumptions of the previous paragraph). In particular, to forestall a likely objection, for every large i, it does not seem that the source of our epistemic inability with respect to the sharp boundary between the $\mathcal{D}^i B$s and the $\neg\mathcal{D}^i B$s is constituted by our *computational* limits in understanding '$\mathcal{D}^i B$': rather, it seems that even suitable finite extensions of us would be in the same epistemic situation as we are with respect to vague expressions we do understand. Since the epistemic inability concerning the sharp boundary between the Bs and the $\neg B$s is due to the vagueness of 'B' (and definitely$^\omega$ so, by the assumptions of the previous paragraph), the epistemic inability concerning the sharp boundary between the $\mathcal{D}^i B$s and the $\neg\mathcal{D}^i B$s is due to the vagueness of '$\mathcal{D}^i B$' (and definitely$^\omega$ so, by closure of the property of being definite$^\omega$ under finite logical consequence, see fn 14). In other words, the vagueness of 'B' is *radical*:

(R) For every i, '$\mathcal{D}^i B$' is definitely$^\omega$ vague.

(R) might however seem to be in contrast with at least one of the considerations adduced in support of (an analogue of) (O), namely that regarding the possibility

[20]With regard to 'B', there certainly is an asymmetry between 0 and 1,000,000 in that 0 is a *limit case* on the scale relevant for the application of 'B' in $\mathbb{S}$, while 1,000,000 is not. However, all the considerations just presented in favour of (O) only rely on 1,000,000's being an *indisputable* (negative) case (in the conversational context presupposed to hold) rather than a *limit* case, so that asymmetry does not seem to be relevant here. Thanks to an anonymous referee for raising this issue.

of predicates like 'ξ is close to 0 and [ξ is far from 1,000,000 or identical with 0]' which force some cases (like 0) to be definitely$^\omega$ positive and some other cases (like 1,000,000) to be definitely$^\omega$ negative. The worry is that the straightforward argument which ensures in this and similar examples that there are definitely$^\omega$ positive and negative cases will also establish such cases as definite sharp boundaries at some higher order or other. Thus, in the case of 'ξ is close to 0 and [ξ is far from 1,000,000 or identical with 0]', it might be natural to think that 0 and 1,000,000 will be, for some i, the definite sharp boundaries between the definitely^i positive cases and the not definitely^i positive cases and between the definitely^i negative cases and the not definitely^i negative cases respectively. Once presented at its strongest for the case of unusual predicates like 'ξ is close to 0 and [ξ is far from 1,000,000 or identical with 0]', the worry can also be applied to the case of usual predicates like '$B\xi$' as a general line of resistance against (R) once (O) has been conceded.

However, focus on the phenomenon of ignorance allows us a conclusive dismissal of the worry. The proponent of the worry grants that 0 is a definitely$^\omega$ positive case. Were vagueness to peter out at the ith order in the manner envisaged, the proponent should thus be able knowledgeably to identify 0 as the sharp boundary between the definitely^i positive cases and the not definitely^i positive cases (for the reason mentioned earlier, we can safely ignore every other possible source of ignorance, such as computational limits). Unfortunately for her, the simple fact is that she isn't. Any identification she might make will sound wholly preposterous, for no minimally serious considerations that could back it up are known (indeed, we even seem to lack a clear conception of what such considerations could consist in).

4 The Paradox

4.1 The Basic Form of the Paradox

We are now in a position to show that, because of the phenomenon of higher-order vagueness as described by (O) and (R), principles of the form of (I) are just as paradoxical as (TOL). Our paradox can most easily be made to emerge by using an appropriate, hopefully self-explanatory, natural-deduction system. We begin with:

1	(1)	$\mathcal{D}^{999,999}\mathcal{D}\neg\mathcal{D}B1,000,000$	(O)
2	(2)	$\mathcal{D}^{999,999}\neg\exists x(\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}Bx')$	(R)
3	(3)	$\mathcal{D}\neg\mathcal{D}B1,000,000$	A
4	(4)	$\neg\exists x(\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}Bx')$	A
5	(5)	$\mathcal{D}\mathcal{D}B999,999$	A
3,5	(6)	$\mathcal{D}\mathcal{D}B999,999 \wedge \mathcal{D}\neg\mathcal{D}B1,000,000$	5,3 $\wedge$-I
3,5	(7)	$\exists x(\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}Bx')$	6 $\exists$-I

3,4,5	(8)	$\bot$	7,4 $\neg$-E
3,4	(9)	$\neg\mathcal{D}\mathcal{D}B999,999$	5,8 $\neg$-I
1,2	(10)	$\mathcal{D}^{999,999}\neg\mathcal{D}\mathcal{D}B999,999$	1,2,3,4,9 $\mathcal{D}$-C,

where the step from (9) to (10) employs the closure rule $\mathcal{D}$-C corresponding to (C) (note that (C) as well as (F) are crucial for (O) to entail (1)). With a qualification I will address in section 4.2, I assume that none of the inference rules used in the paradox can plausibly be rejected. (R) justifies (2) if it is assumed, with the dominant approach, that (second-order) vagueness (definitely^{999,999}) requires (second-order) borderlineness, and that, given suitably iterated definitizations of (T) and of the principle of the form of (M) with '$\mathcal{D}B$' substituted for 'B', by an analogue of theorem 1 (second-order) borderlineness in turn (definitely^{999,999}) requires (second-order) indefiniteness.

Next, we have:

1,2	(1′)	$\mathcal{D}^{999,998}\mathcal{D}\neg\mathcal{D}\mathcal{D}B999,999$	(10)
2′	(2′)	$\mathcal{D}^{999,998}\neg\exists x(\mathcal{D}\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}\mathcal{D}Bx')$	(R)
3′	(3′)	$\mathcal{D}\neg\mathcal{D}\mathcal{D}B999,999$	A
4′	(4′)	$\neg\exists x(\mathcal{D}\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}\mathcal{D}Bx')$	A
5′	(5′)	$\mathcal{D}\mathcal{D}\mathcal{D}B999,998$	A
3′,5′	(6′)	$\mathcal{D}\mathcal{D}\mathcal{D}B999,998 \wedge \mathcal{D}\neg\mathcal{D}\mathcal{D}B999,999$	5′,3′ $\wedge$-I
3′,5′	(7′)	$\exists x(\mathcal{D}\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}\mathcal{D}Bx')$	6′ $\exists$-I
3′,4′,5′	(8′)	$\bot$	7′,4′ $\neg$-E
3′,4′	(9′)	$\neg\mathcal{D}\mathcal{D}\mathcal{D}B999,998$	5′,8′ $\neg$-I
1,2,2′	(10′)	$\mathcal{D}^{999,998}\neg\mathcal{D}\mathcal{D}\mathcal{D}B999,998$	1′,2′,3′,4′,9′ $\mathcal{D}$-C.

(R) justifies (2′) if it is assumed, with the dominant approach, that (third-order) vagueness (definitely^{999,998}) requires (third-order) borderlineness, and that, given suitably iterated definitizations of (T) and of the principle of the form of (M) with '$\mathcal{D}\mathcal{D}B$' substituted for 'B', by an analogue of theorem 1 (third-order) borderlineness in turn (definitely^{999,998}) requires (third-order) indefiniteness. But 999,998 structurally identical arguments eventually lead to '$\neg\mathcal{D}^{1,000,001}B0$', thereby contradicting (O).

This paradox can aptly be called 'a *higher-order sorites paradox*'. Recall from section 2.2 that the standard sorites paradox starts from an extreme which seems to have a certain negative property (the property of being $\neg B$), reasons step by step—using the *prima facie* plausible (TOL)—from a successor case having a certain negative property (indeed, the very same property of the extreme) to the predecessor case also having a certain negative property (indeed, the very same property of the successor case) and concludes that the opposite extreme has a certain negative property (indeed, the very same property of the first extreme) incompatible with a certain positive property it seems to have (the property of being B). Now, in a fully analogous fashion, our paradox too starts from an extreme which seems to have a certain negative property (the property of being $\mathcal{D}^{\omega}\neg B$), reasons step by step—using suitably iterated definitizations of *prima facie* plausible principles of the form of (I)—from a successor case having a certain negative property (for some i and j, the property of being $\mathcal{D}^i\neg\mathcal{D}^jB$) to the predecessor case also having a certain negative property (the property of being $\mathcal{D}^{i-1}\neg\mathcal{D}^{j+1}B$) and concludes that the opposite extreme has a certain negative property (the property of being $\neg\mathcal{D}^{1{,}000{,}001}B$) incompatible with a certain positive property it seems to have (the property of being $\mathcal{D}^{\omega}B$).[21]

The paradox can also be seen as a substantial improvement on similar arguments present in the extant literature. A brief comparison with these is thus in order. The paradoxicality of principles of the form of (I) in the presence of higher-order vagueness (*even without* (O) *and* (R)) has first been argued for by Wright [1987], pp. 262–266; Wright [1991], pp. 141–144, 150–160; Wright [1992], pp. 130–137. However, Wright's argument needs a *prima facie* implausibly strong (**S4**ish) logic for the 'definitely'-operator and has influentially been criticized by Edgington [1993], pp. 193–196; Heck [1993] on these grounds. For the record, I do not find those criticisms conclusive: I have argued in Zardini [2006a] that, for some i, '$\mathcal{D}^iBx \supset \mathcal{D}^{i+1}Bx$' is bound to hold, which means that an **S4**ish logic for the 'definitely'-operator does hold at the ith order (thus opening up the possibility of reinstating an analogue of Wright's argument at that order), and I have argued in Zardini [2006b], pp. 208–213 that i should be identified with 1 (thus reinstating in effect Wright's original argument).[22] Be that as it may with Wright's argument, the same paradoxicality has more recently been argued for by Fara [2003], pp. 196–205, whose argument, however, still needs a logic for the 'definitely'-

[21] In conversation, Patrick Greenough has indicated that a structurally analogous paradox can be produced for *inexact knowledge* rather than *vagueness* (see Williamson [1992] for an influential discussion of inexact knowledge). For concreteness, let the numerals denote the relevant pairwise indiscriminable trees in a series going from 50 ft tall to 10 ft tall, let '$\mathcal{D}\varphi$' be interpreted as 'An ordinary human subject can know by unaided observation that φ' and let '$B\xi$' be interpreted as 'ξ is at least 30 ft tall'. Under this reading, (O) and the suitably iterated definitizations of the relevant (first- and higher-order) borderlineness and indefiniteness principles are still extremely plausible, so that a paradox structurally analogous to our higher-order sorites paradox would arise. The paradox is a genuinely new and independent paradox only insofar as the support for the higher-order borderlineness and indefiniteness principles does not rely on the vagueness of 'An ordinary human subject can know by unaided observation that φ'—in particular, on the vagueness of 'know' (if it does, it is in effect merely an instantiation of the higher-order sorites paradox under a very specific interpretation of '$\mathcal{D}$'). It is very unclear that it does not. Insofar as it does not, it is my view that the solution to this latter paradox will depend on understanding certain crucial features of the nature of reflective knowledge and of closure principles for knowledge. I try to make a start on these issues in **?**.

[22] In Zardini [2006b], p. 214, fn 20, I have also pointed out that, at least given the conditional version of the countable strengthening of (C) mentioned in fn 9, an **S4** logic is bound to hold for '$\mathcal{D}^{\omega}$' (thus opening up the possibility of reinstating an analogue of Wright's argument for '$\mathcal{D}^{\omega}$', see also Wright [2010], pp. 538–539). However, the status of the needed conditional version of the countable strengthening of (C) is moot.

operator acceptable only on some of its specific interpretations (in particular, a logic according to which the controversial rule '$\varphi \vdash \mathcal{D}\varphi$' is valid). Our higher-order sorites paradox, on the contrary, only needs the operator to be factive (by (F)) and closed under finite logical consequence (by (C)), and hence targets also many theories (such as the one offered in Cobreros [2008]) which weaken the logic of 'definitely' in order to cope with the arguments just mentioned.

The paradoxicality of several combinations consisting on the one hand in certain sets of (suitably iterated definitizations of) principles of the form of (I) and on the other hand in certain (suitably iterated definitizations of) '$B0$' and '$\neg B1,000,000$' has recently been argued for by Fine [2008]. However, what I take to be Fine's central result needs the problematic conditional version of the countable strengthening of (C) mentioned in fn 9. Moreover, setting some matters of detail aside, Fine's whole construction needs *reductio* for the background non-modal logic, an inference rule which, whilst both classically and intuitionistically valid, is invalid in most many-valued logics which have seriously been proposed in the vagueness debate. Our higher-order sorites paradox, on the contrary, only needs the 'definitely'-operator to be factive (by (F)) and closed under finite logical consequence (by (C)), and is very easily and naturally generalizable to a wide range of interesting logics in which *reductio* is invalid (as I explain in section 4.2).

The paradoxicality of *margin-for-error* principles of the form of:

(ME) $\neg\exists x(\mathcal{D}Bx \wedge \neg Bx')$

in the presence of higher-order vagueness (*with* (O) *and* (R), or, similarly to what is observed in the final comment of section 4.2, some suitable finite weakening thereof) has been argued for by Gómez-Torrente [1997], pp. 243–245; Gómez-Torrente [2002], pp. 114–117, 119–124; Fara [2002] (to whom Williamson replies in his Williamson [1997], pp. 261–263; Williamson [2002], pp. 144–149). However, margin-for-error principles and their underlying epistemology are very controversial; by (F), they entail but are not entailed by principles of the form of (I), which, on the contrary, seem to be just platitudes about vagueness.

The paradoxicality of principles of the form of (I) in the presence of higher-order vagueness (*with* (O) *and* (R), or, similarly to what is observed in the final comment of section 4.2, some suitable finite weakening thereof) has first been argued for by Zardini [2006a]. However, the quite different argument given in that paper needs some distinctively classical inference rules and a problematic step from *de dicto* definiteness to *de re* definiteness. Our higher-order sorites paradox, on the contrary, is very easily and naturally generalizable to a wide range of interesting logics (as I explain in section 4.2), and only needs the 'definitely'-operator to be factive (by (F)) and closed under finite logical consequence (by (C)). The same paradoxicality has more recently been argued for by Asher et al. [2009], pp. 913–915, who sketch an argument similar to the basic form of the paradox presented in this section (contrary to the previous authors, they only incidentally—in the course of defending a certain version of supervaluationism—touch on their paradox and think that a satisfactory solution to it consists in rejecting suitably iterated definitizations of principles of the form of (I)). However, their argument needs principles of the form of (I) with '$\neg B$' substituted for 'B' (and the ordering inverted), which are problematic on views on which the vagueness of positive predicates like 'B' does not entail the vagueness of negative predicates like '$\neg B$' (or like '$\mathcal{D}\neg B$'). Moreover, their argument only applies to principles strictly of the form of (I). Our higher-order sorites paradox, on the contrary, only needs the vagueness of the positive predicates 'B', '$\mathcal{D}B$', '$\mathcal{D}\mathcal{D}B$' etc., and is very easily and naturally generalizable to a wide range of principles (or rules) similar to but weaker than (I) (as I explain in section 4.2), in such a way as to create a problem for solutions to the paradox that simply consist in rejecting suitably iterated definitizations of principles of the form of (I).

4.2 Weakening the Logic? Weakening Borderlineness or Indefiniteness?

Vagueness naturally prompts doubts about classical logic, so it is incumbent on the proponent of classically valid but logically non-trivial arguments such as the higher-order sorites paradox just presented to give some indications as to whether and how such arguments can be modified so as to count as valid also by the lights of a certain non-classical logic. I have already briefly indicated in section 3.1 how to do so, at least at certain places, in the case of intuitionist and (some) paraconsistent logics. In the vagueness debate, there are of course so many radically different logical approaches that it would be utopic to hope for an argument that is ultimately unobjectionable for all of them. Still, it will prove useful in assessing the real scope of the higher-order sorites paradox just presented to see how it can be modified at least so as to dispense with what is arguably the most problematic inference rule used in it. Moreover, doing so will allow us to see how the paradox can be used also to target borderlineness or indefiniteness principles (or rules) similar to but weaker than (B) or (I).

The most problematic inference rule used in the higher-order sorites paradox just presented is the inference rule of *reductio* (in the steps from (8) to (9), from (8′) to (9′) etc.), which, whilst both classically and intuitionistically valid, is invalid in most many-valued logics which have seriously been proposed in the vagueness debate.[23] *Reductio* has already crucially been used in the last step of the proof of theorem 1. There, we could have appealed instead to *weak contraposition*:

(WC) If $\Gamma, \varphi \vdash^{-} \psi$, then $\Gamma, \neg\psi \vdash \neg\mathcal{D}\varphi$,

where '$\vdash^{-}$' denotes the first-order fragment of $\vdash$ (with the conditional defined as usual) plus (C). (WC) is valid e. g. in the *strong-Kleene* logic $\mathbf{K_3}$ on the most

[23]Thanks to Hartry Field for pressing this worry.

obvious definition of the truth conditions for '$\mathcal{D}$' and should be valid, I submit, on just about any reasonable semantics for the operator. Letting ψ be '$\neg\exists x(\neg\mathcal{D}Bx \wedge \neg\mathcal{D}\neg Bx)$', Γ be {(T), (M)} and φ be '$\exists x(\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$', using (WC) (instead of *reductio*) in the last step of the proof of theorem 1 we can obtain the *weak-indefiniteness* principle:

(WI) $\neg\mathcal{D}\exists x(\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$

(instead of (I)).

We can also establish a useful fact:

Theorem 2. *Under minimal assumptions concerning* $\vdash$*, '*$\neg$*' and '*$\bot$*'* (WC) *is equivalent with* weak reductio*:*

(WR) *If* $\Gamma, \varphi \vdash^{-} \bot$*, then* $\Gamma \vdash \neg\mathcal{D}\varphi$*.*[24]

Proof.

- (WR) *implies* (WC). Suppose that $\Gamma, \varphi \vdash^{-} \psi$. Then, by '$\psi, \neg\psi \vdash^{-} \bot$' and transitivity of $\vdash^{-}$, $\Gamma, \varphi, \neg\psi \vdash^{-} \bot$, and so, by (WR), $\Gamma, \neg\psi \vdash \neg\mathcal{D}\varphi$.
- (WC) *implies* (WR). Suppose that $\Gamma, \varphi \vdash^{-} \bot$. Then, by (WC), $\Gamma, \neg\bot \vdash \neg\mathcal{D}\varphi$, and so, by suppression of logical truths on the left-hand side of $\vdash$, $\Gamma \vdash \neg\mathcal{D}\varphi$.

□

Before presenting how the paradox can then be modified so as to go through with (WI) rather than (I), let me stress that, just as with the application of *reductio* discussed in section 3.1 and contrary to the application of *reductio* occurring in the last step of the proof of theorem 1, the application of *reductio* occurring in the original paradox (in the steps from (8) to (9), from (8′) to (9′) etc.) can be dispensed with (after some fiddling) in favour of the inference rule of *disjunctive syllogism* or variants thereof, which are valid in quite a few many-valued logics in which *reductio* is invalid (e.g. in $\mathbf{K_3}$; a similar remark holds for the eliminability of the application of (WR) in the modified paradox to be presented shortly). To illustrate, in the original paradox we could have inferred '$\neg(\mathcal{D}\mathcal{D}B999,999 \wedge \mathcal{D}\neg\mathcal{D}B1,000,000)$' from (4) by an uncontroversial quantified De Morgan law and universal instantiation, which in turn, together with (3), yields (9) by *modus ponendo tollens* (a variant of disjunctive syllogism, which, working with a negated conjunction rather than a disjunction, allows us to sidestep intuitionist worries). Still, disjunctive syllogism and its variants need not be valid in every reasonable logic of vagueness, and hence it will prove useful to see how the paradox can be modified so as to go through with (WR) rather than *reductio* or disjunctive syllogism and its variants.

Here is how. With (WI) in place, we can run the following variation of the paradox. We begin with:

[24] Thanks to Patrick Greenough for impressing upon me the importance of weak *reductio*.

1	(1)	$\mathcal{D}^{1,999,998}\mathcal{D}\mathcal{D}\neg\mathcal{D}B1,000,000$	(O)
2	(2)	$\mathcal{D}^{1,999,998}\neg\mathcal{D}\exists x(\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}Bx')$	(R)
3	(3)	$\mathcal{D}\mathcal{D}\neg\mathcal{D}B1,000,000$	A
4	(4)	$\neg\mathcal{D}\exists x(\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}Bx')$	A
5	(5)	$\mathcal{D}\mathcal{D}\mathcal{D}B999,999$	A
6	(6)	$\mathcal{D}\neg\mathcal{D}B1,000,000$	A
7	(7)	$\mathcal{D}\mathcal{D}B999,999$	A
6,7	(8)	$\mathcal{D}\mathcal{D}B999,999 \wedge \mathcal{D}\neg\mathcal{D}B1,000,000$	7,6 $\wedge$-I
6,7	(9)	$\exists x(\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}Bx')$	8 $\exists$-I
3,5	(10)	$\mathcal{D}\exists x(\mathcal{D}\mathcal{D}Bx \wedge \mathcal{D}\neg\mathcal{D}Bx')$	3,5,6,7,9 $\mathcal{D}$-C
3,4,5	(11)	$\bot$	10,4 $\neg$-E
3,4	(12)	$\neg\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}B999,999$	5,11 W$\neg$-I
1,2	(13)	$\mathcal{D}^{1,999,998}\neg\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}B999,999$	1,2,3,4,12 $\mathcal{D}$-C,

where the step from (11) to (12) employs the weak-*reductio* rule W$\neg$-I corresponding to (WR). (R) justifies (2) if it is assumed, with the dominant approach, that (second-order) vagueness (definitely^{1,999,998}) requires (second-order) borderlineness, and that, given suitably iterated definitizations of (T) and of the principle of the form of (M) with '$\mathcal{D}B$' substituted for 'B', by an analogue of the (WC)-version of theorem 1 (second-order) borderlineness in turn (definitely^{1,999,998}) requires (second-order) weak indefiniteness.

Next, we have:

$1'$	$(1')$	$\mathcal{D}^{1,999,996}\mathcal{D}\mathcal{D}\neg\mathcal{D}^4B999,999$	(13)
$2'$	$(2')$	$\mathcal{D}^{1,999,996}\neg\mathcal{D}\exists x(\mathcal{D}\mathcal{D}^4Bx \wedge \mathcal{D}\neg\mathcal{D}^4Bx')$	(R)
$3'$	$(3')$	$\mathcal{D}\mathcal{D}\neg\mathcal{D}^4B999,999$	A
$4'$	$(4')$	$\neg\mathcal{D}\exists x(\mathcal{D}\mathcal{D}^4Bx \wedge \mathcal{D}\neg\mathcal{D}^4Bx')$	A
$5'$	$(5')$	$\mathcal{D}\mathcal{D}\mathcal{D}^4B999,998$	A
$6'$	$(6')$	$\mathcal{D}\neg\mathcal{D}^4B999,999$	A
$7'$	$(7')$	$\mathcal{D}\mathcal{D}^4B999,998$	A
$6'$,$7'$	$(8')$	$\mathcal{D}\mathcal{D}^4B999,998 \wedge \mathcal{D}\neg\mathcal{D}^4B999,999$	$7'$,$6'$ $\wedge$-I
$6'$,$7'$	$(9')$	$\exists x(\mathcal{D}\mathcal{D}^4Bx \wedge \mathcal{D}\neg\mathcal{D}^4Bx')$	$8'$ $\exists$-I
$3'$,$5'$	$(10')$	$\mathcal{D}\exists x(\mathcal{D}\mathcal{D}^4Bx \wedge \mathcal{D}\neg\mathcal{D}^4Bx')$	$3'$,$5'$,$6'$,$7'$,$9'$ $\mathcal{D}$-C
$3'$,$4'$,$5'$	$(11')$	$\bot$	$10'$,$4'$ $\neg$-E
$3'$,$4'$	$(12')$	$\neg\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}^4B999,998$	$5'$,$11'$ W$\neg$-I
$1'$,$2'$	$(13')$	$\mathcal{D}^{1,999,996}\neg\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}^4B999,998$	$1'$,$2'$,$3'$,$4'$,$12'$ $\mathcal{D}$-C.

(R) justifies $(2')$ if it is assumed, with the dominant approach, that (fifth-order) vagueness (definitely^{1,999,996}) requires (fifth-order) borderlineness, and that, given suitably iterated definitizations of (T) and of the principle of the form of (M) with '$\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}B$' substituted for '$B$', by an analogue of the (WC)-version of theorem 1 (fifth-order) borderlineness in turn (definitely^{1,999,996}) requires (fifth-order) weak indefiniteness. But 999,998 structurally identical arguments eventually lead to '$\neg\mathcal{D}^{3,000,001}B0$', thereby contradicting (O).

Notice that the application of (WR) in a proof of (WI) along the lines of the proof of theorem 1 should be relevantly (and hence paraconsistently) unexceptionable, exactly for the same reasons why the application of *reductio* in the proof of theorem 1 should. Notice also, however, that (WR) in its full strength—and its application in the modified paradox (in the steps from (11) to (12), from (11′) to (12′) etc.)—is paraconsistently (and hence relevantly) invalid, exactly for the same reasons why *reductio* in its full strength—and its application in the original paradox (in the steps from (8) to (9), from (8′) to (9′) etc.)—is. Again, analogously to the discussion in section 3.1, additional, plausible assumptions concerning intensional connections between x''s being $\mathcal{D}\mathcal{D}\neg\mathcal{D}^iB$ and x's being $\neg\mathcal{D}^{i+3}B$ (licensing the inference from the one to the other and spelling out in effect the established fact that x is not definitely a definite sharp boundary for $\mathcal{D}^iB$ness) would have to be made in order to turn the argument into a relevantly (and hence paraconsistently) valid one.

Might not a defender of the dominant approach weaken the modality and only maintain the weaker claim that, if an expression is vague, it is not definitely the case that it does not present borderline cases (instead of the usual, stronger claim that, if an expression is vague, it presents borderline cases), thereby endorsing the *weak-borderlineness* principle:

(WB) $\neg\mathcal{D}\neg\exists x(\neg\mathcal{D}Bx \wedge \neg\mathcal{D}\neg Bx)$

(instead of the usual, stronger (B))? No. For plausible logical assumptions suffice to establish:

Theorem 3. (WB) *entails* '$\neg\mathcal{D}\mathcal{D}\exists x(\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$'*.*

Proof. As shown in the proof of theorem 1, '$\exists x(\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$' entails '$\neg\exists x(\neg\mathcal{D}Bx \wedge \neg\mathcal{D}\neg Bx)$', and so, by (C), '$\mathcal{D}\exists x(\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$' entails '$\mathcal{D}\neg\exists x(\neg\mathcal{D}Bx \wedge \neg\mathcal{D}\neg Bx)$', wherefore, by (WC), (WB) entails '$\neg\mathcal{D}\mathcal{D}\exists x(\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$'.

□

And '$\neg\mathcal{D}\mathcal{D}\exists x(\mathcal{D}Bx \wedge \mathcal{D}\neg Bx')$', by a slight modification of the paradox just presented, can also be shown to lead to contradiction.

Relatedly, I have sometimes heard it say that, in response to the paradox, one should switch from the indefiniteness *principle* (I) to the weaker indefiniteness *rules*:

(IR^0) $\mathcal{D}Bx \vdash \neg\mathcal{D}\neg Bx'$;

(IR^1) $\mathcal{D}\neg Bx' \vdash \neg\mathcal{D}Bx$.

It is true that such indefiniteness rules may not imply (WI). However, plausible logical assumptions suffice to establish:

Theorem 4. (O)*,* (IR^0) *and* (WR) *imply* (WB)*.*

Proof. Consider an arbitrarily chosen object a. Suppose that a is $\mathcal{D}B$. Then, by (IR^0), a' is $\neg\mathcal{D}\neg B$. Suppose for weak *reductio* that it is not the case that there are borderline Bs. Then, by an uncontroversial quantified De Morgan law and universal instantiation, it is not the case that a' is both $\neg\mathcal{D}B$ and $\neg\mathcal{D}\neg B$. But a' is $\neg\mathcal{D}\neg B$, and so, by *modus ponendo tollens*, a' is $\neg\neg\mathcal{D}B$, wherefore, by double-negation elimination, a' is $\mathcal{D}B$. Hence, under the supposition that a is $\mathcal{D}B$, one can infer that a' is also $\mathcal{D}B$. However, by (O), 0 is $\mathcal{D}B$, and so, since a was arbitrary, by 1,000,000 applications of the above argument it follows that $1,000,000$ is $\mathcal{D}B$. Contradiction with (O). Therefore, by (WR), (WB) follows.

□

And, given theorem 3, (WB) suffices to reinstate the paradox.

In closing, let me stress that inspection of our higher-order sorites paradox immediately reveals that the transfinite strength of (O) and (R) is wholly dispensable. For once the finite length of the relevant soritical series[25] has been fixed, only weakenings of (O) and (R) where '$\mathcal{D}^\omega$' is replaced by sufficiently large *finite* concatenations of '$\mathcal{D}$' (and where the initial quantifier in (R) is restricted to a sufficiently large initial segment of canonical numerals) are needed to breed paradox. Thus, even if the considerations adduced in support of (O) and (R) should prove resistible, the defender of the dominant approach is still faced with the daunting task of showing that no suitable finite weakenings of (O) and (R) are ever available for any soritical series—that indisputable positive and negative cases are much less definitely so than we thought them to be and that higher-order vagueness is much less definitely radical than we thought it to be. I for one cannot see how the defender of the dominant approach has in the present state of information a guarantee that that task can be accomplished in its full generality. At best, agnosticism about the matter would seem to be mandated.[26]

5 Conclusion

Our higher-order sorites paradox seems then to show suitably iterated definitizations of higher-order principles of the form of (I) (and, therefore, of higher-order principles of the form of (B)), put forward by the dominant approach in order to characterize, for every i, the nature of the vagueness of '$\mathcal{D}^i B$', to be inconsistent with suitably iterated definitizations of 0's being B and of $1,000,000$'s being $\neg B$. The situation is thus fully analogous to the one faced by the naive theory of vagueness, where the original sorites paradox apparently showed (TOL), put forward by the naive theory in order to characterize the nature of the vagueness of 'B', to

[25]A *soritical series* is any series of objects which, like $\mathbb{S}$, can induce a sorites paradox.

[26]Thanks to Crispin Wright for emphasizing to me the dialectical importance of this point.

be inconsistent with 0's being B and $1,000,000$'s being $\neg B$. Hence, keeping fixed the highly plausible principles (O) and (R) (or suitable finite weakenings thereof), the dominant approach loses its main advantage over the naive theory—its alleged immunity to any form of sorites paradox. In any event, it is not known not to do so, given that a suitable combination of these principles is at least not known not to be true. All this invites looking for a new diagnosis of the sorites paradox.[27]

References

Nicholas Asher, Josh Dever, and Chris Pappas. Supervaluations debugged. *Mind*, 118:901–933, 2009.

David Barnett. Does vagueness exclude knowledge? *Philosophy and Phenomenological Research*, 82:22–45, 2011.

Pablo Cobreros. Supervaluationism and logical consequence: A third way. *Studia Logica*, 90:291–312, 2008.

Cian Dorr. Vagueness without ignorance. *Philosophical Perspectives*, 17:83–113, 2003.

Cian Dorr. Iterated definiteness. In Richard Dietz and Sebastiano Moruzzi, editors, *Cuts and Clouds: Essays on the Nature and Logic of Vagueness*, pages 550–575. Oxford University Press, Oxford, 2010.

Dorothy Edgington. Wright and Sainsbury on higher-order vagueness. *Analysis*, 53:193–200, 1993.

Delia Graff Fara. Shifting sands: An interest-relative theory of vagueness. *Philosophical Topics*, 28:45–81, 2000.

Delia Graff Fara. An anti-epistemicist consequence of margin for error semantics for knowledge. *Philosophy and Phenomenological Research*, 64:127–142, 2002.

Delia Graff Fara. Gap principles, penumbral consequence, and infinitely higher-order vagueness. In JC Beall, editor, *Liars and Heaps*, pages 195–221. Oxford University Press, Oxford, 2003.

[27] If I believed in borderline cases as conceived of by the dominant approach, I would block our higher-order sorites paradox using one of the logics developed in Zardini [2008]; Zardini [2011a], suitably extended with a 'definitely'-operator. What would fail in such logics is the unrestricted transitivity of the consequence relation implicitly relied on in sections 4.1, 4.2. These logics, however, are explicitly designed to be such that, in them, (TOL) itself is consistent (and is indeed a conservative extension of a background mathematical theory), and hence turning to them would in effect obliterate any advantage the dominant approach might have had over the naive theory with respect to immunity to any form of sorites paradox.

Hartry Field. Disquotational truth and factually defective discourse. *The Philosophical Review*, 103:405–452, 1994.

Hartry Field. No fact of the matter. *Australasian Journal of Philosophy*, 81: 457–480, 2003.

Hartry Field. Solving the paradoxes, escaping revenge. In JC Beall, editor, *Revenge of the Liar*, pages 78–144. Oxford University Press, Oxford, 2007.

Hartry Field. *Saving Truth from Paradox*. Oxford University Press, Oxford, 2008.

Kit Fine. Vagueness, truth and logic. *Synthese*, 30:265–300, 1975.

Kit Fine. The impossibility of vagueness. *Philosophical Perspectives*, 22:111–136, 2008.

Mario Gómez-Torrente. Two problems for an epistemicist view of vagueness. *Philosophical Issues*, 8:237–245, 1997.

Mario Gómez-Torrente. Vagueness and margin for error principles. *Philosophy and Phenomenological Research*, 64:107–125, 2002.

Patrick Greenough. Vagueness: A minimal theory. *Mind*, 112:235–281, 2003.

Patrick Greenough. Contextualism about vagueness and higher-order vagueness. *Proceedings of the Aristotelian Society Supplementary Volume*, 79:167–190, 2005.

Richard Heck. A note on the logic of (higher-order) vagueness. *Analysis*, 53: 201–208, 1993.

Dominic Hyde. From heaps and gaps to heaps of gluts. *Mind*, 106:641–660, 1997.

Vann McGee. *Truth, Vagueness, and Paradox*. Hackett, Indianapolis, 1991.

Vann McGee and Brian McLaughlin. Distinctions without a difference. *Southern Journal of Philosophy Supplement*, 33:204–251, 1995.

Diana Raffman. Vagueness without paradox. *The Philosophical Review*, 103:41–74, 1994.

David Sanford. Borderline logic. *American Philosophical Quarterly*, 12:29–39, 1975.

Stephen Schiffer. *The Things We Mean*. Oxford University Press, Oxford, 2003.

Roy Sorensen. *Blindspots*. Clarendon Press, Oxford, 1988.

Paula Sweeney and Elia Zardini. Vagueness and practical interest. In Paul Égré and Nathan Klinedinst, editors, *Vagueness and Language Use*, pages 249–282. Palgrave MacMillan, Basingstoke, 2011.

Michael Tye. Vague objects. *Mind*, 99:535–557, 1990.

Peter Unger. There are no ordinary things. *Synthese*, 41:117–154, 1979.

Timothy Williamson. Inexact knowledge. *Mind*, 101:217–242, 1992.

Timothy Williamson. *Vagueness*. Routledge, London, 1994.

Timothy Williamson. Replies to commentators. *Philosophical Issues*, 8:255–265, 1997.

Timothy Williamson. Epistemicist models: Comments on Gómez-Torrente and Graff. *Philosophy and Phenomenological Research*, 64:143–150, 2002.

Crispin Wright. Further reflections on the sorites paradox. *Philosophical Topics*, 15:227–290, 1987.

Crispin Wright. The sorites paradox and its significance for the interpretation of semantic theory. In Neil Cooper and Pascal Engel, editors, *New Inquiries Into Meaning and Truth*, pages 135–162. Harvester Wheatsheaf, Hemel Hempsted, 1991.

Crispin Wright. Is higher-order vagueness coherent? *Analysis*, 52:129–139, 1992.

Crispin Wright. On being in a quandary: Relativism, vagueness, logical revisionism. *Mind*, 60:45–98, 2001.

Crispin Wright. The illusion of higher-order vagueness. In Richard Dietz and Sebastiano Moruzzi, editors, *Cuts and Clouds: Essays on the Nature and Logic of Vagueness*, pages 523–549. Oxford University Press, Oxford, 2010.

Elia Zardini. Squeezing and stretching: How vagueness can outrun borderlineness. *Proceedings of the Aristotelian Society*, 106:419–426, 2006a.

Elia Zardini. Higher-order vagueness and paradox: The glory and misery of S4 definiteness. In Jurğis Šķilters, Matti Eklund, Ólafur Páll Jónsson, and Olav Wiegand, editors, *Paradox: Logical, Cognitive and Communicative Aspects*, volume I of *The Baltic International Yearbook of Cognition, Logic and Communication*, pages 203–220. University of Latvia Press, Riga, 2006b.

Elia Zardini. A model of tolerance. *Studia Logica*, 90:337–368, 2008.

Elia Zardini. First-order tolerant logics. *The Review of Symbolic Logic*, 2011a. Forthcoming.

Elia Zardini. Seconde naïveté. ms, 2011b.

Elia Zardini. Inexact knowledge, positive introspection, and closure. ms, 2011c.

Elia Zardini. Red and orange. ms, 2011d.