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A Model of Tolerance

Abstract. According to the naive theory of vagueness, the vagueness of an expression consists in the existence of both positive and negative cases of application of the expression and in the non-existence of a sharp cut-off point between them. The sorites paradox shows the naive theory to be inconsistent in most logics proposed for a vague language. The paper explores the prospects of saving the naive theory by revising the logic in a novel way, placing principled restrictions on the transitivity of the consequence relation. A lattice-theoretical framework for a whole family of (0th-order) "tolerant logics" is proposed and developed. Particular care is devoted to the relation between the salient features of the formal apparatus and the informal logical and semantic notions they are supposed to model. A suitable non-transitive counterpart to classical logic is defined. Some of its properties are studied, and it is eventually shown how an appropriate regimentation of the naive theory of vagueness is consistent in such a logic.

Keywords: lattice theory, many-valued logics, sorites paradox, transitivity of consequence, vagueness.

1. Introduction and Overview

Some men are bald: a man with absolutely no hairs on his scalp is bald. Other men are not bald: a man with a full head of hairs is not bald. It is also very intuitive that the difference between the bald and the non-bald is not an exquisitely fine difference: if a man is bald, so is one with just one more hair on his scalp. In other words, it is also very intuitive that the expression 'bald' is *tolerant*, in the sense that one-hair differences do not make a difference to its positive or negative application.[1] The theory

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[1] 'Bald', as many other vague expressions, is arguably *multi-dimensional*: the correctness of its application to a particular item depends on the item's location on at least two distinct dimensions of comparison. In our case, for example, the correctness of an application of 'bald' to a man depends not only on the number of hairs on his scalp, but also on their distribution, density, thickness etc. Because of multi-dimensionality, the principle that, if a man is bald, so is one with just one more hair on his scalp, is, strictly speaking, false: a man with a sufficient number i of hairs, widely distributed, homogeneously dense and appropriately thick may well count as non-bald, whereas a man with $i-1$ hairs so poorly distributed as to cover only a square centimeter of his scalp, so heterogeneously dense as to leave a hairless circle in the middle, so thin as to be invisible will count as bald. Therefore, tolerance principles apply straightforwardly only to

based on the union of these or relevantly similar claims may well be called 'the *naive theory of vagueness*'. It is a *theory of vagueness* insofar as its trio of claims can be taken to give an account of what the vagueness of an expression consists in—very roughly, in its being tolerant whilst having both positive and negative cases of application.[2] It is a *naive* theory of vagueness insofar as its *prima facie* theoretical advantages of simplicity, explanatory power and preservation of ordinary intuitions (see [15]) are cast into grave doubt by an argument—the so-called '*sorites paradox*'—purporting to show its inconsistency, in a way similar to that in which the naive theory of sets, based on the unrestricted set-comprehension schema, is affected by the set-theoretical paradoxes.

An informal presentation of the sorites paradox goes as follows. Consider the premises:

(1) A man with 0 hairs is bald;

(2) A man with 1,000,000 hairs is not bald;

(3) If a man with i hairs is bald, so is a man with $i + 1$ hairs.

The naive theory of vagueness is committed to all these three premises. However, from (3) we have that, if a man with 0 hairs is bald, so is a man with 1 hair, which, together with (1), yields that a man with 1 hair is bald. Yet, from (3) we also have that, if a man with 1 hair is bald, so is a man with 2 hairs, which, together with the previous lemma that a man with 1 hair is bald, yields that a man with 2 hairs is bald. With another 999,997 structurally identical arguments, we reach the conclusion that a man with 999,999 hairs is bald. From (3) we also have that, if a man with 999,999 hairs is bald, so is a man with 1,000,000 hairs, which, together with the previous lemma that a man with 999,999 hairs is bald, yields that a man with 1,000,000 hairs is bald. It would then seem that the contradictory of (2) follows simply from (1) and (3).[3]

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uni-dimensional expressions—in the case of multi-dimensional ones, a tolerance principle applies on a particular dimension of comparison only under the assumption that the values of the other dimensions are held constant. For simplicity's sake, I will henceforth ignore multi-dimensionality.

[2]Such a theory thus stands in sharp contrast to the dominant approach in contemporary theories of vagueness, guided by the hypothesis that the vagueness of an expression consists in its having *borderline cases* of application (I offer structural criticisms of the dominant approach as being liable to higher-order sorites paradoxes in [12]; [13]; [14]).

[3]In the simple form presented in the text, the argument has to make use of 999,999 applications of the *contraction* rule as well (see [1, pp. 523–6]). However, clumsier versions

I think that all the three characteristic claims of the naive theory of vagueness are *in fact* true and jointly provide a satisfactory account of what the vagueness of an expression consists in, but I will not argue directly for that here (I have done so in [15]). Rather, I will focus here on the problem presented by the sorites paradox to the naive theory of vagueness, offering the basics of a logical framework in which the theory *can* be true. As we will see, the basic idea of the logical framework is not that of weakening the properties of any particular logical constant (i.e. weakening the *operational rules* of the logic). While this might be desirable in other respects, it won't help much in saving the naive theory of vagueness from inconsistency, since, as we have seen, the sorites paradox need only use the rule of *modus ponens* for the conditional used to formulate the tolerance principles, and rejection of this rule—in addition of course to being in itself highly implausible—would seem to deprive those principles of their intended *force* (which consists in *not allowing any difference* in, say, baldness between two men who only differ of 1 in the number of hairs on their scalp). The basic idea of the logical framework is rather that of weakening the properties of the consequence relation itself (i.e. weakening the *structural rules* of the logic), in particular the *transitivity* property.[4]

No doubt vagueness has offered other reasons, apparently independent of the sorites paradox, to question the validity of certain classical laws and rules concerning specific logical constants: the phenomenon of borderline cases has for instance induced many a commentator to reject the law of excluded middle (see e.g. [3]). I will thus start by describing a very minimal logical basis which, while being non-transitive, does not prejudge any of the other many interesting issues arising in the philosophy of the logic of vagueness. I will call any logic which is definable on this basis and in which the naive theory of vagueness is consistent owing to the failure of the transitivity property 'a *tolerant logic*'. From such a basis, I will then build up to my favoured specific system (mirroring, in a non-transitive setting, classical logic), indicating along the way how other weaker non-transitive systems (mirroring, in a non-transitive setting, some or other non-classical logic) can be defined. An appropriate justification of such preference lies outside the scope of this paper: suffice it to say that I regard classical patterns of reasoning as unchallenged by vagueness insofar as they are compatible with a

of it are readily available which only employ 1,000,000 different instances of (3) instead of (3) itself, and so do not require contraction (against what Copeland seems to imply).

[4]The proposal of restricting the transitivity of logical consequence calls of course for an extended philosophical discussion of both its viability and import I cannot hope to undertake here. I try to make some progress on these issues in [**?**].

suitable weakening of the transitivity of the consequence relation (for e.g. a defense of the law of excluded middle see [17]).[5]

The framework is thus *neutral* in the sense that it can fruitfully be employed also by a theorist who, while attracted to the naive theory of vagueness, has also reason to reject some other classical pattern of reasoning for vague expressions. Moreover, as mentioned in fn 2, most contemporary theories of vagueness, requiring borderline cases for an expression whenever the expression is vague, are also subject to (higher-order) sorites paradoxes. These paradoxes too can be blocked in a principled way by restricting the transitivity of the consequence relation. Therefore, such theories too should find the present framework congenial: its neutrality will allow them to focus on (one of) the particular logic(s) which is generated by restricting the transitivity of their originally favoured consequence relation.

The rest of the paper is organized as follows. Section 2 provides and explains the general many-valued semantic framework of tolerant logics. Building on this basic framework, section 3 stepwise develops a tolerant counterpart to classical logic and investigates some of its properties. Section 4 discusses how the notion of consistency is best modelled in the present framework and offers a proof of the consistency in a tolerant logic of the sentential fragment of the naive theory of vagueness. Section 5 draws the conclusions which follow from the consistency proof for the wider vagueness debate and hints at some of the outstanding philosophical and technical issues.

2. A Neutral Framework

2.1. Syntax

The core idea of tolerant logics—the failure of the transitivity property for the consequence relation—already emerges at the sentential level. Furthermore, a significant fragment of the naive theory of vagueness can be adequately regimented in a standard sentential language. Hence, in this paper, we will focus on the study of such a language, the vague zeroth-order language $\mathscr{L}^0$. To stress, it is not that $\mathscr{L}^0$ is said to be vague because some of its expressions are vague: being a formal language, its non-logical atomic expressions are not interpreted, and so *a fortiori* not vague. Moreover, there is no plausibility in the idea that the mathematically precise semantics which will be provided for its logical expressions should induce any vagueness in

[5] Arguably, this preference for classicality should be qualified in an important respect once tolerant logics are extended to the first-order level (see [18]).

them. Rather, $\mathscr{L}^0$ is said to be vague because the logic determined by its semantics is such that an adequate regimentation of a significant fragment of the naive theory of vagueness is consistent in it. I should also emphasize that the *metalanguage* $\mathscr{M}$ within which we will conduct our study of $\mathscr{L}^0$ will be assumed to be classical (in particular, assumed to be such that its consequence relation is transitive), and that the *metatheory* used in $\mathscr{M}$ will be the classical set theory ZFC.[6]

DEFINITION 2.1. The set $AS_{\mathscr{L}^0}$ of the *atomic symbols* of $\mathscr{L}^0$ is defined by enumeration as follows:[7]

- The denumerable set $VAR_{\mathscr{L}^0}$ of variables $P_0, P_1, P_2 \ldots, Q_0, Q_1, Q_2 \ldots, R_0, R_1, R_2 \ldots$ is a subset of $AS_{\mathscr{L}^0}$;
- The 1ary sentential operator $\neg$ belongs to $AS_{\mathscr{L}^0}$;
- The 2ary sentential operators $\wedge$, $\vee$ and $\rightarrow$ belong to $AS_{\mathscr{L}^0}$;
- (and) belong to $AS_{\mathscr{L}^0}$;
- Nothing else belongs to $AS_{\mathscr{L}^0}$.

Note that, given the generality of the framework we will be developing, we cannot assume the definability of any of the standard sentential operators in terms of others. In particular, in order to let non-transitivity emerge already at the very first stages of the construction, it will prove useful to have a primitive conditional $\rightarrow$ bearing a privileged connection to the consequence relation.

DEFINITION 2.2. The set $WFF_{\mathscr{L}^0}$ of *well-formed formulae* (wffs) of $\mathscr{L}^0$ (equivalently, given that $\mathscr{L}^0$ is zeroth-order, the set of sentences of $\mathscr{L}^0$) can be defined by recursion in the usual way:[8]

- If $\varphi \in VAR_{\mathscr{L}^0}$, $\varphi \in WFF_{\mathscr{L}^0}$;
- If $\varphi \in WFF_{\mathscr{L}^0}$, $(\neg\varphi) \in WFF_{\mathscr{L}^0}$;

[6] The use of a classical metalanguage in the explanation of a non-classical object language is of course one of the cruces of any proposal of deviation from classical logic. I cannot hope to address here the philosophical issues related to this asymmetry nor the crucial philosophical and technical question as to whether and how much of an explanation of a non-classical object language can be provided using a metalanguage with the same logic.

[7] Throughout, formal symbols are understood autonymously to refer to themselves.

[8] Throughout, 'φ', 'ψ' and 'χ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over $WFF_{\mathscr{L}^0}$.

- If $\varphi, \psi \in WFF_{\mathscr{L}^0}$, $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in WFF_{\mathscr{L}^0}$;
- Nothing else belongs to $WFF_{\mathscr{L}^0}$.

Henceforth, to save on brackets, I will assume the usual scope hierarchy among the operators (with $\neg$ binding more strongly than $\wedge$ and $\vee$, and with these in turn binding more strongly than $\rightarrow$) and left associativity of the 2ary operators (so that $(\varphi_0 \star \varphi_1 \star \varphi_2 \ldots \star \varphi_i)$ reads $(((\ldots(\varphi_0 \star \varphi_1) \star \varphi_2) \ldots \star \varphi_i)$, with $\star$ being a 2ary operator). I will also drop the outermost brackets of a self-standing wff.

2.2. Tolerant Semantic Structures

Definition 2.3. We will take a *sequence* of wffs of $\mathscr{L}^0$ to be a function whose domain is some suitable initial segment of the ordinals and whose range is a subset of $WFF_{\mathscr{L}^0}$.[9] As usual:[10]

- $\Gamma, \Delta := \Gamma \bigcup \{\langle \delta_1, \varphi \rangle$: for some $\delta_0 \in \mathrm{dom}(\Delta), \delta_1 = \mathrm{dom}(\Gamma) + \delta_0$ and $\varphi = \Delta(\delta_0)\}$;
- $\Gamma, \varphi := \Gamma, \langle \varphi \rangle$,

where ':=' is a metalinguistic symbol expressing the definition relation and $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are, respectively, the domain and the range of R.

Definition 2.4. A *logic* $\mathbf{L}$ for $\mathscr{L}^0$ is any subset of $\{\langle \Gamma_0, \Gamma_1 \rangle : \mathrm{ran}(\Gamma_0), \mathrm{ran}(\Gamma_1) \subseteq WFF_{\mathscr{L}^0}\}$.[11]

As anticipated, we start our semantic construction by developing a minimal logical basis, the basic tolerant logic $\mathbf{B}^0$, which already displays the core idea of any tolerant logic—the failure of the transitivity property for the consequence relation—but which is otherwise completely neutral with regard to the issues concerning the philosophy of the logic of a vague language. By adding further and further constraints on the semantics, we will then be able to specify stronger and stronger tolerant logics. We will make use of standard lattice-theoretical semantics, introducing the modifications appropriate for obtaining failures of the transitivity property for the consequence relation.

[9] Every function is assumed to be total unless otherwise specified.

[10] Throughout, 'Γ', 'Δ', 'Θ', 'Λ', 'Ξ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over the set of sequences whose range is included in $WFF_{\mathscr{L}^0}$.

[11] We will thus be working in a multiple-conclusion framework, this being required in order to achieve the desired generality.

DEFINITION 2.5. A $\mathbf{B}^0$-*structure* $\mathfrak{S}$ for $\mathscr{L}^0$ is a 5ple $\langle V_\mathfrak{S}, D_\mathfrak{S}, \preceq_\mathfrak{S}, \mathrm{tol}_\mathfrak{S}, O_\mathfrak{S}\rangle$, where:

- $V_\mathfrak{S}$ is a non-empty set of objects (the "values");
- $D_\mathfrak{S}$ is a non-empty subset of $V_\mathfrak{S}$ (the "designated values") such that:

  (D_0) For every $v_0, v_1 \in V_\mathfrak{S}$, if $v_0 \in D_\mathfrak{S}$ and $v_0 \preceq_\mathfrak{S} v_1$, then $v_1 \in D_\mathfrak{S}$ (see the next item for a definition of $\preceq_\mathfrak{S}$)

  ($D_\mathfrak{S}$ is an *upper set*);
- $\preceq_\mathfrak{S}$ is a partial ordering (reflexive, anti-symmetric, transitive relation) on $V_\mathfrak{S}$ such that:

  (glb/lub$^2_0$) For every $v_0, v_1 \in V_\mathfrak{S}$, $\{v_0, v_1\}$ has a greatest lower bound (glb) and a least upper bound (lub)

  ($\preceq_\mathfrak{S}$ thus corresponds to a *lattice*);
- $\mathrm{tol}_\mathfrak{S}$ is a "tolerance" function from $V_\mathfrak{S}$ into pow($V_\mathfrak{S}$) (the powerset of $V_\mathfrak{S}$) such that:

  (tol$_0$) For every $v \in V_\mathfrak{S}$, $v \in \mathrm{tol}_\mathfrak{S}(v)$;

  (tol$_1$) For every $v \in V_\mathfrak{S}$, $\mathrm{tol}_\mathfrak{S}(v)$ is an upper set.

  Note in particular that $\mathrm{tol}_\mathfrak{S}(v)$ is allowed to contain values which are *not* contained in the upper set whose minimum element is v. As we will see, this "tolerating" feature of tol is crucial in generating failures of the transitivity property for the consequence relation;
- $O_\mathfrak{S}$ is a non-empty set of operations on $V_\mathfrak{S}$. In particular, $\{\mathrm{neg}_\mathfrak{S}, \mathrm{impl}_\mathfrak{S}\} \subseteq O_\mathfrak{S}$, where:

  (neg$^{\Rightarrow}_0$) For every $v_0, v_1 \in V_\mathfrak{S}$, if $v_0 \preceq_\mathfrak{S} v_1$, then $\mathrm{neg}_\mathfrak{S}(v_1) \preceq_\mathfrak{S} \mathrm{neg}_\mathfrak{S}(v_0)$;[12]

  (neg$^{\Rightarrow}_1$) For every $v \in V_\mathfrak{S}$, $v \preceq_\mathfrak{S} \mathrm{neg}_\mathfrak{S}(\mathrm{neg}_\mathfrak{S}(v))$;

  (impl$^{\Rightarrow}_0$) For every $v_0, v_1 \in V_\mathfrak{S}$, if $v_1 \in \mathrm{tol}_\mathfrak{S}(v_0)$, then $\mathrm{impl}_\mathfrak{S}(v_0, v_1) \in D_\mathfrak{S}$;

  (impl$^{\Leftarrow}_0$) For every $v_0, v_1 \in V_\mathfrak{S}$, if $\mathrm{impl}_\mathfrak{S}(v_0, v_1) \in D_\mathfrak{S}$, then $v_1 \in \mathrm{tol}_\mathfrak{S}(v_0)$.

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[12]Note that (neg$^{\Rightarrow}_0$) and (neg$^{\Rightarrow}_1$) can be neatly packaged into the so-called "*law of intuitionist contraposition*" that, for every $v_0, v_1 \in V_\mathfrak{S}$, if $v_0 \preceq_\mathfrak{S} \mathrm{neg}_\mathfrak{S}(v_1)$, then $v_1 \preceq_\mathfrak{S} \mathrm{neg}_\mathfrak{S}(v_0)$.

Again, note in particular how $\mathrm{impl}_{\mathfrak{S}}$ relates to $\mathrm{tol}_{\mathfrak{S}}$, and especially how ($\mathrm{impl}_0^{\Leftarrow}$) allows $\mathrm{impl}_{\mathfrak{S}}(v_0, v_1)$ to belong to $D_{\mathfrak{S}}$ even if v_1 does not belong to the upper set whose minimum element is v_0.[13] Henceforth, we will focus on the case where $\{\mathrm{neg}_{\mathfrak{S}}, \mathrm{impl}_{\mathfrak{S}}\} = O_{\mathfrak{S}}$, but it is clear how, given the rich structure generated by $V_{\mathfrak{S}}$, $D_{\mathfrak{S}}$, $\preceq_{\mathfrak{S}}$ and $\mathrm{tol}_{\mathfrak{S}}$, many other interesting operations may be defined and added to $O_{\mathfrak{S}}$ (and be expressed by corresponding operators in some extension of $\mathscr{L}^0$).[14] In order to exploit the full power of ($\mathrm{neg}_0^{\Rightarrow}$), we place another constraint on $\mathrm{tol}_{\mathfrak{S}}$:

(tol_2) For every v_0, $v_1 \in V_{\mathfrak{S}}$, if $v_1 \in \mathrm{tol}_{\mathfrak{S}}(v_0)$, then $\mathrm{neg}_{\mathfrak{S}}(v_0) \in \mathrm{tol}_{\mathfrak{S}}(\mathrm{neg}_{\mathfrak{S}}(v_1))$.

A $\mathbf{B}^0$-structure can then be used to interpret $\mathscr{L}^0$ once it is equipped with an interpretation function for $VAR_{\mathscr{L}^0}$ and once suitable recursive clauses for the sentential operators are given.

DEFINITION 2.6. A $\mathbf{B}^0$-*model* $\mathfrak{M}$ for $\mathscr{L}^0$ based on a $\mathbf{B}^0$-structure $\mathfrak{S}$ is a 6ple $\langle V_{\mathfrak{M}}, D_{\mathfrak{M}}, \preceq_{\mathfrak{M}}, \mathrm{tol}_{\mathfrak{M}}, O_{\mathfrak{M}}, \mathrm{int}_{\mathfrak{M}}\rangle$, where $V_{\mathfrak{M}}$, $D_{\mathfrak{M}}$, $\preceq_{\mathfrak{M}}$, $\mathrm{tol}_{\mathfrak{M}}$ and $O_{\mathfrak{M}}$ are identical to $V_{\mathfrak{S}}$, $D_{\mathfrak{S}}$, $\preceq_{\mathfrak{S}}$, $\mathrm{tol}_{\mathfrak{S}}$ and $O_{\mathfrak{S}}$ respectively, and $\mathrm{int}_{\mathfrak{M}} : VAR_{\mathscr{L}^0} \mapsto V_{\mathfrak{M}}$ is an interpretation function for $VAR_{\mathscr{L}^0}$.

DEFINITION 2.7. $\mathrm{int}_{\mathfrak{M}}$ is extended to a full valuation function $\mathrm{val}_{\mathfrak{M}} : WFF_{\mathscr{L}^0} \mapsto V_{\mathfrak{M}}$ by the following recursion:

($\mathrm{val}_{VAR_{\mathscr{L}^0}}$) If $\varphi \in VAR_{\mathscr{L}^0}$, $\mathrm{val}_{\mathfrak{M}}(\varphi) = \mathrm{int}_{\mathfrak{M}}(\varphi)$;

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[13] We have defined $\mathrm{impl}_{\mathfrak{S}}$ in terms of $\mathrm{tol}_{\mathfrak{S}}$. Alternatively, and as an an anonymous referee pointed out to me, we could have defined $\mathrm{tol}_{\mathfrak{S}}$ in terms of $\mathrm{impl}_{\mathfrak{S}}$ by saying that $v_1 \in \mathrm{tol}_{\mathfrak{S}}(v_0)$ iff $\mathrm{impl}_{\mathfrak{S}}(v_0, v_1) \in D_{\mathfrak{S}}$. Imposing natural constraints on $\mathrm{impl}_{\mathfrak{S}}$, the resulting metatheory would be equivalent to the metatheory proposed in the text (with both ($\mathrm{impl}_0^{\Rightarrow}$) and ($\mathrm{impl}_0^{\Leftarrow}$)). On conceptual grounds, I prefer the latter, since it characterizes $\mathrm{tol}_{\mathfrak{S}}$ (which, as we will see, is crucial in the definition of a non-transitive consequence relation) independently of particular logical operations such as $\mathrm{impl}_{\mathfrak{S}}$ (indeed, it uses $\mathrm{tol}_{\mathfrak{S}}$ to define $\mathrm{impl}_{\mathfrak{S}}$ and could use it to define other interesting logical operations, as in fn 14).

[14] In particular, given the characteristic "lowering" behaviour of the conjunction operation (to be specified shortly), it may prove useful to define an equivalence operation $\mathrm{equiv}_{\mathfrak{S}}$ such that:

($\mathrm{equiv}_0^{\Rightarrow}$) For every $v_0, v_1 \in V_{\mathfrak{S}}$, if $v_1 \in \mathrm{tol}_{\mathfrak{S}}(v_0)$ and $v_0 \in \mathrm{tol}_{\mathfrak{S}}(v_1)$, then $\mathrm{equiv}_{\mathfrak{S}}(v_0, v_1) \in D_{\mathfrak{S}}$;

($\mathrm{equiv}_0^{\Leftarrow}$) For every $v_0, v_1 \in V_{\mathfrak{S}}$, if $\mathrm{equiv}_{\mathfrak{S}}(v_0, v_1) \in D_{\mathfrak{S}}$, then $v_1 \in \mathrm{tol}_{\mathfrak{S}}(v_0)$ and $v_0 \in \mathrm{tol}_{\mathfrak{S}}(v_1)$,

and extend $\mathscr{L}^0$ with a new primitive, biconditional-like, 2ary operator $\leftrightarrow$ expressing $\mathrm{equiv}_{\mathfrak{S}}$. Save for a brief remark later (fn 27) substantiating this point, in this paper I will not pursue further the investigation of the logic of such $\leftrightarrow$.

($\mathrm{val}_\neg$) $\mathrm{val}_\mathfrak{M}(\neg\varphi) = \mathrm{neg}_\mathfrak{M}(\mathrm{val}_\mathfrak{M}(\varphi))$;

($\mathrm{val}_\wedge$) $\mathrm{val}_\mathfrak{M}(\varphi \wedge \psi) = \mathrm{glb}_\mathfrak{M}(\{\mathrm{val}_\mathfrak{M}(\varphi), \mathrm{val}_\mathfrak{M}(\psi)\})$;

($\mathrm{val}_\vee$) $\mathrm{val}_\mathfrak{M}(\varphi \vee \psi) = \mathrm{lub}_\mathfrak{M}(\{\mathrm{val}_\mathfrak{M}(\varphi), \mathrm{val}_\mathfrak{M}(\psi)\})$;

($\mathrm{val}_\rightarrow$) $\mathrm{val}_\mathfrak{M}(\varphi \rightarrow \psi) = \mathrm{impl}_\mathfrak{M}(\mathrm{val}_\mathfrak{M}(\varphi), \mathrm{val}_\mathfrak{M}(\psi))$.

Determining which value $v \in V_\mathfrak{M}$ a wff has in $\mathfrak{M}$, $\mathrm{val}_\mathfrak{M}$ *a fortiori* determines whether $v \in D_\mathfrak{M}$ or not—in other words, $\mathrm{val}_\mathfrak{M}$ determines whether the value of a wff is designated or not. Now, in standard many-valued semantics, the role played by designated values can be (informally) explained as follows. It is assumed that the *actual* semantics of an interpreted language $\mathscr{J}$ whose logical properties one is interested in studying exhibits at least the *general features* of the semantics used in the mathematical study of a formal language $\mathscr{K}$ into which $\mathscr{J}$ can be adequately regimented. For example, $\mathscr{J}$ might be a fragment of English expressing first-order Peano-Dedekind arithmetic and $\mathscr{K}$ be a standard formal first-order language (with identity and functors): then just as, in every model of $\mathscr{K}$, every sentence (closed wff) of $\mathscr{K}$ is assigned either 1 or 0 as value (but not both),[15] so it is assumed that every sentence of $\mathscr{J}$ is either true or false (but not both). If the formal semantics of $\mathscr{K}$ is many-valued, there will typically be, included in the set of all the values a sentence can be assigned in a model, a set of designated values. What do such values correspond to in the semantics of $\mathscr{J}$?

The usual answer in philosophy of logic to this question is, roughly, that they correspond to the "*good*" values a sentence can have—that is, those which make a sentence good enough to be asserted, good enough to be believed, good enough to be acted upon etc. (see [6, p. 216]).[16] At least in this respect, then, what designated values correspond to plays the same role in a many-valued framework as *truth* plays in a two-valued framework, for, in a two-valued framework, it is truth that which warrants assertion, belief and action.

This is the place to enter a crucial clarification concerning the present

[15] Of course, even if usual, the particular choice of 1 and 0 to model truth and falsity is completely conventional—any two other objects recognized by the background mathematical theory will do.

[16] "*Semantically*" good enough. For it may well be that other non-semantic features (e. g. epistemic ones) contribute to the determination of a sentence's assertability, believability, enactability etc. A similar qualification concerning truth should be understood below as implicit. I don't mean the qualification to be enforced for the "levels of goodness" to be introduced shortly.

use of a many-valued semantics.[17] Such a semantics is here used with the main purpose of inducing a certain (family of) logic(s). The different values are supposed to model the different levels of goodness a sentence can have in terms of its assertability, believability, enactability etc., where the notion of a level of goodness can be reduced for simple predications to the position of the relevant object in the ordering generated by the contextually relevant dimension of comparison. The values assigned to compound sentences by the semantics are supposed to model the way in which we understand the level of goodness of a compound sentence to be determined by the level of goodness of its components. Emphatically, the different intermediate levels of goodness are not different ways in which a sentence can be neither true nor false (and so neither do the extreme levels of goodness—if they exist—coincide with truth and falsity), nor do levels of goodness represent an ordering of truth among sentences (on this very last point I thus diverge from the interpretation of the lattice-theoretical many-valued framework offered by [10]). The truth about truth is that a sentence 'P' is true iff P, and false otherwise—and this is manifestly too simple a notion to use if one is trying to develop a non-classical logic in a classical metatheory. It is levels of goodness, understood in the minimalist way just explained, which allow us to draw the fine-grained distinctions required by non-classical reasoning.[18]

Let me note that understanding logical operations as operating on the fine structure of levels of goodness rather than simply on truth and falsity does not mean of course that they are not significantly constrained by truth and falsity—indeed, they should be such that the logic they generate is not too strong as to rule out intuitively possible assignments of truth and falsity and not too weak as to allow intuitively impossible assignments of truth and falsity. Let me also note that an additional layer of complexity is induced by the fact that talk of 'good values' should presumably be vague, so that our use of a classical metalanguage risks to misrepresent what good values are. This problem connects with some of the issues mentioned in fn 6—here I will only add that the risk is at least partially averted by the quantification over models in the definition of the consequence relation and by the rejection

[17]Thanks to Graham Priest and to an anonymous referee for urging consideration of this issue.

[18]I thus agree with Michael Dummett's strictures against those who "reduce the semantic notion of logical consequence to a purely algebraic tool" ([2, p. 293]), but disagree with him when he claims that "[o]n the assumption that all our sentences possess determinate truth-values, there is simply nothing that one can think of that a truth-table would leave unexplained concerning the meaning of the sentential operator for which it was correct" ([2, p. 294]).

that one of the classically described models is the intended model of a vague language. Let me finally stress that I am painfully aware that a much more detailed discussion of these issues would be needed in order to make my claims persuasive, but that I hope that what I have just said gives enough indication as to how to understand the framework to be developed.

In a two-valued framework, truth also plays a crucial role in the definition of the consequence relation: the informal characterization of consequence as *necessary truth preservation from the premises to the conclusions* gets formally translated as preservation of 1 from the premises to the conclusions in every model. In a many-valued framework, it is then very natural to define consequence as preservation of designated value from the premises to the conclusions (in the sense that, if every premise has a designated value in a model $\mathfrak{M}$, then some conclusion also has a designated value in $\mathfrak{M}$). It is thus guaranteed that, when one argues validly from good premises, one will reach some good conclusion.

This is the point of entry of the crucial modification I would like to propose in order to generate failures of the transitivity property for the consequence relation. Consider again the tolerance function tol:[19] informally, it implements the idea that, if v_0 counts as *very good* a value for a sentence to have, any $v_1 \in \mathrm{tol}(v_0)$ will also count as *good enough* a value for a sentence to have (to be asserted, believed, acted upon etc.). Of course, since we are working in a classical (and thus transitive) metalanguage, we cannot require that, if v_0 counts as *good enough* a value for a sentence to have, any $v_1 \in \mathrm{tol}(v_0)$ will likewise count as *good enough* a value for a sentence to have, for such a principle would breed paradox (given the fact that it need not be the case that, for every $v_0, v_1, v_2 \in V$, if $v_2 \in \mathrm{tol}(v_1)$ and $v_1 \in \mathrm{tol}(v_0)$, then $v_2 \in \mathrm{tol}(v_0)$).[20] Still, it might reasonably be argued that our inferential practices with a vague language lend support to the idea that all consequence in a vague language guarantees is that, when one argues validly from very good premises, one will reach some good enough conclusion. I will not attempt here to establish this claim in its generality—rather, I will briefly try to support its plausibility by illustrating how it is supposed to work in a particular case.

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[19]Henceforth, I will drop subscripts for models and structures if no ambiguity threatens.

[20]Letting $\mathrm{tol}^* = \{\langle X, Y\rangle : X \subseteq V$ and, for every v_0, $v_0 \in Y$ iff, for some $v_1 \in X$, $v_0 \in \mathrm{tol}(v_1)\}$, the point in the text can be put by saying that tol^* is *non-idempotent*, in the sense that it need not be the case that, for every $X \subseteq V$, $\mathrm{tol}^*(\mathrm{tol}^*(X)) = \mathrm{tol}^*(X)$ (indeed, tol^* is *non-subidempotent*, in the sense that it need not be the case that, for every $X \subseteq V$, $\mathrm{tol}^*(\mathrm{tol}^*(X)) \subseteq \mathrm{tol}^*(X)$).

(4) A man with i hairs is bald

might have a very good value, because it has been accepted on the basis of perception, or intuition, or testimony ultimately relying on either perception or intuition etc.

(5) If a man with i hairs is bald, then a man with $i+1$ is bald

might also have a very good value, because it belongs to the naive theory of vagueness. Given this, it is reasonable to expect that:

(6) A man with $i+1$ hairs is bald,

which presumably follows from (5) and (4), will have a good enough value, and thus that it will also be assertable, believable, enactable etc. However, it is not equally reasonable to expect that, because:

(7) If a man with $i+1$ hairs is bald, then a man with $i+2$ is bald

has also a very good value (since it too belongs to the naive theory of vagueness),

(8) A man with $i+2$ hairs is bald,

which presumably follows from (7) and (6), will also have a good enough value: for (6) only has a good enough value, and it is not equally reasonable to expect that a sentence having a merely good enough value can always be fed as a premise into an inferential process to yield a conclusion with a good enough value (as it is the case, on the contrary, when every premise has not just a good enough value, but a very good value).[21]

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[21]If (6) were also knowable by, say, perception, there would be no ground for denying that (6) actually has a very good value (rather than a merely good enough value), and so it would seem that there would be no ground for denying that (8) will also have a good enough value. Correspondingly, at the level of logic, if (6) were also knowable by perception, then (7) would license the inference from (6) to (8) simply in virtue of *modus ponens* (rather than in virtue of (4), (5), *modus ponens* and transitivity), and so it would seem that (8) should be accepted. Is the view I am developing thus committed to holding that, although a man with $i+1$ hairs is bald, this fact cannot be known by perception, but only by inference using tolerance principles? It is not. If baldness goes together with knowability by perception, then 'knowable by perception to be bald' will be just as vague as 'bald', and so a tolerant logic should be used when reasoning with it (and with other expressions related to it like 'has a very good value'). In such a logic, one cannot

It seems to me that the description of the case I have given agrees with those which would be given by many speakers if subjected to this short stretch of a soritical series. Typically, they would accept (4), (5) and (7) (and (6)) without feeling thereby compelled to accept (8), even though they would accept the validity of both *modus-ponens* arguments (note that, typically, they would not feel compelled to accept (8) even if there were no evidence that it is false). Typically, they would justify this complex pattern of acceptance by claiming that, even though in itself valid, the argument in question (i.e. *modus ponens*) should not be "pushed too far". In order to explain these reactions, it is plausible to conjecture that the underlying implicit conception of validity is such that it is only thought to guarantee that very good premises yield a good enough conclusion, so that it need not be thought to guarantee that a double application of *modus ponens* starting from very good premises will ultimately issue in a good enough conclusion (as the case just described makes clear). On this conception—which is the key to the failure of transitivity—validity is not a matter of *preservation* of anything (neither of being very good from the premises to the conclusions nor of being good enough from the premises to the conclusions), but a matter of *connection* between the premises' being very good and the conclusions' being good enough.[22]

---

infer from the original description of the situation that (8) will also have a good enough value (or, correspondingly, that (8) should be accepted), even if one assumes that baldness goes together with knowability by perception. For so to infer would require an invalid transitivity step in the language talking about knowability by perception and matters related to it. One can only get so far as to accept that (6) actually has a very good value (in addition to (7)'s having a very good value and to (6) and (7)'s entailing (8)), or, correspondingly, that (7) would license the inference (6) to (8) simply in virtue of *modus ponens* (in addition to (6)'s being true), but one cannot infer from this that (8) will also have a good enough value, or, correspondingly, that (8) should be accepted (just like one can only get so far as to accept (6) and (7), but cannot infer (8) from this). I discuss these issues at greater length in [16].

[22] I take it that the distinction between very good values and good enough values has some very intuitive appeal and import in our evaluation of reasons: we would ordinarily distinguish between one's reasons to accept a certain sentence being so strong as to permit (or even mandate) acceptance of whatever follows from that sentence and one's reasons to accept a certain sentence being simply strong enough as to permit (or even mandate) acceptance of that sentence. In the former case, one's reasons allow (or even mandate) one to take the sentence as a starting point for further reasoning, whereas in the latter case they only allow (or even mandate) one to take the sentence as a terminal point of acceptance (see [8, pp. 196–9] for a defence of this distinction within a transitive framework). It should go without saying that the distinction also makes perfectly good probabilistic sense. I dwell more on it in [16].

2.3. Basic Tolerant Logic

Now, our semantic structures give us all the resources to capture the distinction between very good values and good enough values and to deploy it in order to define a consequence relation whose hallmark is the guarantee that very good premises lead to good enough conclusions.

DEFINITION 2.8. The set of "tolerated values" $T_{\mathfrak{S}}$ of a structure $\mathfrak{S}$ is defined as follows:

(T) $T_{\mathfrak{S}} := \bigcup_{d \in D_{\mathfrak{S}}} \mathrm{tol}(d)$

We can then let the set of designated values $D_{\mathfrak{S}}$ represent the very good values of $\mathfrak{S}$, and let the set of tolerated values $T_{\mathfrak{S}}$ represent the good enough values of $\mathfrak{S}$.

THEOREM 2.9. *T is an upper set.*

PROOF. Suppose that $v_0 \in T$ and $v_0 \preceq v_1$. Then, for some $d \in D$, $v_0 \in \mathrm{tol}(d)$, and so is such that, for every v_2, if $v_0 \preceq v_2$, $v_2 \in \mathrm{tol}(d)$ as well. Therefore, since $v_0 \preceq v_1$, $v_1 \in \mathrm{tol}(d)$ as well, and so $v_1 \in T$ as well.

■

DEFINITION 2.10. The consequence relation on pairs of sequences of wffs $\in WFF_{\mathscr{L}^0}$ constituting the basic tolerant logic $\mathbf{B}^0$ ($\models_{\mathbf{B}^0}$) is defined as follows:

($\models_{\mathbf{B}^0}$) A sequence of wffs Δ is a $\models_{\mathbf{B}^0}$-consequence of a sequence of wffs Γ ($\Gamma \models_{\mathbf{B}^0} \Delta$) iff, for every $\mathbf{B}^0$-model $\mathfrak{M}$, if, for every $\varphi \in \mathrm{ran}(\Gamma)$, $\mathrm{val}_{\mathfrak{M}}(\varphi) \in D_{\mathfrak{M}}$, then, for some $\psi \in \mathrm{ran}(\Delta)$, $\mathrm{val}_{\mathfrak{M}}(\psi) \in T_{\mathfrak{M}}$.

It's easy to check that $\mathbf{B}^0$ exhibits some basic *structural properties* (note that full associativity is already guaranteed by our choice of sequences as terms of the consequence relation):

THEOREM 2.11. $\mathbf{B}^0$ *exhibits the following structural properties:*[23]

(C^l) *If* $\Gamma_0 \vdash \Delta$, *then, for every* Γ_1 *such that* $\mathrm{fld}(\Gamma_0) = \mathrm{fld}(\Gamma_1)$, $\Gamma_1 \vdash \Delta$*;*

(C^r) *If* $\Gamma \vdash \Delta_0$, *then, for every* Δ_1 *such that* $\mathrm{fld}(\Delta_0) = \mathrm{fld}(\Delta_1)$, $\Gamma \vdash \Delta_1$*;*

(W^l) *If* $\Gamma, \varphi, \varphi \vdash \Delta$, *then* $\Gamma, \varphi \vdash \Delta$*;*

(W^r) *If* $\Gamma \vdash \Delta, \varphi, \varphi$, *then* $\Gamma \vdash \Delta, \varphi$*;*

[23] We state structural properties in a general fashion, using a variable '$\vdash$' taking as values specific consequence relations (such as $\models_{\mathbf{B}^0}$). $\mathrm{fld}(R)$ is the field of R.

(K^l) *If* $\Gamma \vdash \Delta$*, then* $\Gamma, \varphi \vdash \Delta$*;*

(K^r) *If* $\Gamma \vdash \Delta$*, then* $\Gamma \vdash \Delta, \varphi$*;*

(I) *If* Γ *is non-empty,*[24] $\Gamma \vdash \Gamma$.

PROOF.

- (C^l), (C^r): Immediate from the field invariance of the sequences.
- (W^l), (W^r): Immediate from the fact that $\{\psi : \psi \in \text{ran}(\Gamma, \varphi, \varphi)\} = \{\psi : \psi \in \text{ran}(\Gamma, \varphi)\}$.
- (K^l): Immediate from the fact that $\{\psi : \psi \in \text{ran}(\Gamma)\} \subseteq \{\psi : \psi \in \text{ran}(\Gamma, \varphi)\}$.
- (K^r): Immediate from the fact that $\{\psi : \psi \in \text{ran}(\Delta)\} \subseteq \{\psi : \psi \in \text{ran}(\Delta, \varphi)\}$.
- (I): If Γ is non-empty and, for every $\varphi \in \text{ran}(\Gamma)$, $\text{val}(\varphi) \in D$, then, for some $\varphi \in \text{ran}(\Gamma)$, $\text{val}(\varphi) \in D$, and so *a fortiori* $\text{val}(\varphi) \in T$ (if Γ is empty, then, even though, for every $\varphi \in \text{ran}(\Gamma)$, $\text{val}(\varphi) \in D$, there is no $\varphi \in \text{ran}(\Gamma)$ such that $\text{val}(\varphi) \in T$).

■

The failure of a particular structural property such as transitivity can thus be achieved in full autonomy from the other usual structural properties.[25] It is too seldom noticed that transitivity itself comes in two flavours, left and right. *Left transitivity* concerns, roughly, the legitimacy of chaining a class of inferences together with another inference once some of the conclusions of the class are jointly sufficient to constitute a component of the *premises* (left part) of the latter inference:

(T^l) If, for every $\varphi \in \text{ran}(\Theta)$, $\Gamma \vdash \Delta, \varphi$ and $\Lambda, \Theta \vdash \Xi$, then $\Lambda, \Gamma \vdash \Delta, \Xi$.

---

[24] Note that, given our official stipulations about sequences (definition 2.3), the empty sequence just is the empty set $\varnothing$.

[25] In the few previous attempts to develop a non-transitive consequence relation, this has not always been so. For example, in the non-transitive system mooted in [7, pp. 238–43], (W^l) fails. The philosophical motivation for the system is quite different from that for tolerant logics, as it rather concerns issues of relevance. For such issues, weakenings of other structural rules might well be desirable, as witnessed also by the system proposed by [9, pp. 185–200, 253–65] (belonging to the same tradition as Smiley's), in which (K^l)—in addition to transitivity—fails.

Right transitivity concerns, roughly, the legitimacy of chaining an inference together with a class of inferences once some of the premises of the class are jointly sufficient to constitute a component of the *conclusions* (right part) of the former inference, and is obtained by dualizing consequence with entailment in (T^l):

(T^r) If $\Xi \vdash \Lambda, \Theta$ and, for every $\varphi \in \mathrm{ran}(\Theta)$, $\Delta, \varphi \vdash \Gamma$, then $\Delta, \Xi \vdash \Lambda, \Gamma$.

Given (W^l) and (W^r), (T^l) implies the *cumulative left-transitivity property*:

(CT^l) If, for every $\varphi \in \mathrm{ran}(\Theta)$, $\Gamma \vdash \Delta, \varphi$ and $\Gamma, \Theta \vdash \Delta$, then $\Gamma \vdash \Delta$,

and, given (K^l) and (K^r), it is implied by it. Analogously, given (W^l) and (W^r), (T^r) implies the *cumulative right-transitivity property*:

(CT^r) If $\Delta \vdash \Gamma, \Theta$ and, for every $\varphi \in \mathrm{ran}(\Theta)$, $\Delta, \varphi \vdash \Gamma$, then $\Delta \vdash \Gamma$,

and, given (K^l) and (K^r), it is implied by it.

Given that $\models_{\mathbf{B}^0}$ satisfies all the other structural properties needed in order for (CT^l) and (CT^r) to entail (T^l) and (T^r) respectively, the counterexamples we will provide to (T^l) and (T^r) will be counterexamples to (CT^l) and (CT^r) as well. A first kind of counterexample to (T^l) and (T^r) will emerge by studying further properties of $\mathbf{B}^0$ relating to specific operators.

Some elementary rules and laws for the sentential operators are already valid in $\mathbf{B}^0$. For $\rightarrow$, we have from ($\mathrm{impl}_0^{\Leftarrow}$):

THEOREM 2.12. *The rule of* modus ponens*:*

($\mathrm{MP}_{\rightarrow}$) $\varphi \rightarrow \psi, \varphi \models_{\mathbf{B}^0} \psi$

is valid.

PROOF. Suppose that $\mathrm{val}(\varphi \rightarrow \psi), \mathrm{val}(\varphi) \in D$. Then, by ($\mathrm{impl}_0^{\Leftarrow}$), $\mathrm{val}(\psi) \in \mathrm{tol}(\mathrm{val}(\varphi))$, and so, since $\mathrm{val}(\varphi) \in D$, $\mathrm{val}(\psi) \in T$.

■

Crucially, ($\mathrm{impl}_0^{\Leftarrow}$), validating ($\mathrm{MP}_{\rightarrow}$), already suffices to trigger failures of the transitivity properties for $\mathbf{B}^0$:

THEOREM 2.13. $\models_{\mathbf{B}^0}$ *satisfies neither* (T^l) *nor* (T^r)*.*

Proof.

- (T^l): Given ($\mathrm{MP}_\rightarrow$), $P_0 \rightarrow Q_0, P_0 \models_{\mathbf{B}^0} Q_0$ and $Q_0 \rightarrow R_0, Q_0 \models_{\mathbf{B}^0} R_0$. Setting $\Gamma = P_0 \rightarrow Q_0, P_0$, $\Delta = \varnothing$, $\Theta = Q_0$, $\Lambda = Q_0 \rightarrow R_0$ and $\Xi = R_0$, (T^l) yields that $Q_0 \rightarrow R_0, P_0 \rightarrow Q_0, P_0 \models_{\mathbf{B}^0} R_0$, which is false, since there are $\mathbf{B}^0$-models where $\mathrm{val}(P_0) \in D$, $\mathrm{val}(Q_0) \in \mathrm{tol}(\mathrm{val}(P_0))$ (and so $\mathrm{val}(P_0 \rightarrow Q_0) \in D$) but $\notin D$, $\mathrm{val}(R_0) \in \mathrm{tol}(\mathrm{val}(Q_0))$ (and so $\mathrm{val}(Q_0 \rightarrow R_0) \in D$) but $\notin T$.
- (T^r): Again, given ($\mathrm{MP}_\rightarrow$), $P_0 \rightarrow Q_0, P_0 \models_{\mathbf{B}^0} Q_0$ and $Q_0 \rightarrow R_0, Q_0 \models_{\mathbf{B}^0} R_0$. Setting $\Xi = P_0 \rightarrow Q_0, P_0$, $\Lambda = \varnothing$, $\Theta = Q_0$, $\Delta = Q_0 \rightarrow R_0$ and $\Gamma = R_0$, (T^r) yields that $Q_0 \rightarrow R_0, P_0 \rightarrow Q_0, P_0 \models_{\mathbf{B}^0} R_0$, which is false as explained above.

■

That such a failure emerges with $\rightarrow$ should not be surprising, given the privileged connection of $\rightarrow$ with the consequence relation $\models_{\mathbf{B}^0}$ (connection determined by the role played by tol in the specification of both ($\mathrm{impl}_0^{\Leftarrow}$) and ($\models_{\mathbf{B}^0}$)).[26]

Turning to the interaction of this operator with $\neg$, we note that, thanks to ($\mathrm{impl}_0^{\Rightarrow}$), ($\mathrm{impl}_0^{\Leftarrow}$) and ($\mathrm{tol}_2$), the novelty of the framework does not interfere with the validity of the intuitionistically acceptable rule of contraposition:

Theorem 2.14. *The rule of* contraposition*:*

(CONTR) $\varphi \rightarrow \psi \models_{\mathbf{B}^0} \neg\psi \rightarrow \neg\varphi$

is valid.

Proof. Suppose that $\varphi \rightarrow \psi \in D$. Then, by ($\mathrm{impl}_0^{\Leftarrow}$), $\mathrm{val}(\psi) \in \mathrm{tol}(\mathrm{val}(\varphi))$. Therefore, by ($\mathrm{tol}_2$), $\mathrm{neg}(\mathrm{val}(\varphi)) \in \mathrm{tol}(\mathrm{neg}(\mathrm{val}(\psi)))$, and so, by ($\mathrm{impl}_0^{\Rightarrow}$), $\mathrm{val}(\neg\psi \rightarrow \neg\varphi) \in D$.[27]

■

---

[26]In more standard many-valued logics, like the fuzzy logics introduced by [4]; [11], one can define a conditional such that its designatedness allows for small drops in value from the antecedent to the consequent (see [5] for a suggestion along these lines). Such a conditional would of course fail to be transitive, and could be used to formulate consistent tolerant principles. However, in a standard many-valued framework such a conditional would also crucially fail to satisfy *modus ponens* (since *modus ponens* does not unrestrictedly preserve designated value for the conditional in question), which I regard as a betrayal of the spirit of tolerance (see section 1).

[27]Related to a comment made earlier (fn 14), we can now appreciate why an equivalence operator $\leftrightarrow$ defined in terms of equiv would not be equivalent to the conjunction of condi-

For $\wedge$ and $\vee$, we have from (D_0):

THEOREM 2.15. *The rules of* simplification *and* addition*:*

(SIMP$_0$) $\varphi \wedge \psi \models_{\mathbf{B}^0} \varphi$;

(SIMP$_1$) $\varphi \wedge \psi \models_{\mathbf{B}^0} \psi$;

(ADD$_0$) $\varphi \models_{\mathbf{B}^0} \varphi \vee \psi$;

(ADD$_1$) $\psi \models_{\mathbf{B}^0} \varphi \vee \psi$

are valid.

PROOF. Immediate from (D_0).

■

Interestingly, we do have from (D_0) and theorem 2.9 the following restricted transitivity properties for $\mathbf{B}^0$:

THEOREM 2.16. *The properties of* conjunction in the premises *and* disjunction in the conclusions*:*

(CP$_0$) *If* $\Gamma, \varphi \models_{\mathbf{B}^0} \Delta$, *then* $\Gamma, \varphi \wedge \psi \models_{\mathbf{B}^0} \Delta$;

(CP$_1$) *If* $\Gamma, \psi \models_{\mathbf{B}^0} \Delta$ *then* $\Gamma, \varphi \wedge \psi \models_{\mathbf{B}^0} \Delta$;

(DC$_0$) *If* $\Gamma \models_{\mathbf{B}^0} \Delta, \varphi$ *then* $\Gamma \models_{\mathbf{B}^0} \Delta, \varphi \vee \psi$;

(DC$_1$) *If* $\Gamma \models_{\mathbf{B}^0} \Delta, \psi$ *then* $\Gamma \models_{\mathbf{B}^0} \Delta, \varphi \vee \psi$

hold.

PROOF.

- (CP$_0$): Suppose that $\Gamma, \varphi \models_{\mathbf{T}} \Delta$, that, for every $\chi_0 \in \mathrm{ran}(\Gamma)$, $\mathrm{val}(\chi_0) \in D$ and that $\mathrm{val}(\varphi \wedge \psi) \in D$. Then, by (D_0), $\mathrm{val}(\varphi) \in D$, and so, since $\Gamma, \varphi \models_{\mathbf{T}} \Delta$, for some $\chi_1 \in \mathrm{ran}(\Delta)$, $\mathrm{val}(\chi_1) \in T$.
- (CP$_1$): Analogous.
- (DC$_0$): Suppose that $\Gamma \models_{\mathbf{B}^0} \Delta, \varphi$ and that, for every $\chi_0 \in \mathrm{ran}(\Gamma)$, $\mathrm{val}(\chi_0) \in D$. Then, since $\Gamma \models_{\mathbf{B}^0} \Delta, \varphi$, either, for some $\chi_1 \in \mathrm{ran}(\Delta)$, $\mathrm{val}(\chi_1) \in T$ or $\mathrm{val}(\varphi) \in T$. If the latter, by theorem 2.9, $\mathrm{val}(\varphi \vee \psi) \in T$.

---

tionals $(\varphi \to \psi) \wedge (\psi \to \varphi)$. Consider a model $\mathfrak{M}$ such that $\mathrm{val}_{\mathfrak{M}}(\varphi) = v_0$, $\mathrm{val}_{\mathfrak{M}}(\psi) = v_1$ and $v_0 \in \mathrm{tol}_{\mathfrak{M}}(v_1)$ and $v_1 \in \mathrm{tol}_{\mathfrak{M}}(v_0)$. Then, by $(\mathrm{equiv}_0^{\Rightarrow})$, $\mathrm{val}_{\mathfrak{M}}(\varphi \leftrightarrow \psi) = \mathrm{equiv}_{\mathfrak{M}}(v_0, v_1) = v_2 \in D_{\mathfrak{M}}$. Moreover, by $(\mathrm{impl}_0^{\Rightarrow})$, $\mathrm{val}_{\mathfrak{M}}(\varphi \to \psi) = \mathrm{impl}_{\mathfrak{M}}(v_0, v_1) = v_3 \in D_{\mathfrak{M}}$ and $\mathrm{val}_{\mathfrak{M}}(\psi \to \varphi) = \mathrm{impl}_{\mathfrak{M}}(v_1, v_0) = v_4 \in D_{\mathfrak{M}}$. However, there is no guarantee that $\mathrm{val}_{\mathfrak{M}}((\varphi \to \psi) \wedge (\psi \to \varphi)) = \mathrm{glb}(\{v_3, v_4\}) = v_5 \in D_{\mathfrak{M}}$, let alone that $v_5 = v_2$.

- (DC_1): Analogous.

■

Turning to the interaction of these operators with $\neg$, we note that, thanks to ($neg_0^{\Rightarrow}$) and ($neg_1^{\Rightarrow}$), the novelty of the general framework does not interfere with the validity of the intuitionistically acceptable De Morgan rules:

THEOREM 2.17. *The* De Morgan *rules:*

(DM_0) $\varphi \vee \psi \models_{\mathbf{B}^0} \neg(\neg\varphi \wedge \neg\psi)$*;*

(DM_1) $\varphi \wedge \psi \models_{\mathbf{B}^0} \neg(\neg\varphi \vee \neg\psi)$*;*

(DM_2) $\neg(\varphi \vee \psi) \models_{\mathbf{B}^0} \neg\varphi \wedge \neg\psi$

are valid.

PROOF.

- (DM_0): By ($val_\wedge$), $val(\neg\varphi \wedge \neg\psi) = glb(\{val(\neg\varphi), val(\neg\psi)\})$, and so, by ($neg_0^{\Rightarrow}$), $val(\neg\neg\varphi) \preceq val(\neg(\neg\varphi \wedge \neg\psi))$ and $val(\neg\neg\psi) \preceq val(\neg(\neg\varphi \wedge \neg\psi))$, wherefore, by ($neg_1^{\Rightarrow}$), $val(\varphi) \preceq val(\neg(\neg\varphi\wedge\neg\psi))$ and $val(\psi) \preceq val(\neg(\neg\varphi \wedge \neg\psi))$. But, by ($val_\vee$), $val(\varphi \vee \psi) = lub(\{val(\varphi), val(\psi)\})$, and so $val(\varphi \vee \psi) \preceq val(\neg(\neg\varphi \wedge \neg\psi))$.
- (DM_1): By ($val_\wedge$), $val(\varphi\wedge\psi) = glb(\{val(\varphi), val(\psi)\})$, and so, by ($neg_0^{\Rightarrow}$), $val(\neg\varphi) \preceq val(\neg(\varphi \wedge \psi))$ and $val(\neg\psi) \preceq val(\neg(\varphi \wedge \psi))$. But, by ($val_\vee$), $val(\neg\varphi\vee\neg\psi) = lub(\{val(\neg\varphi), val(\neg\psi)\})$, and so $val(\neg\varphi\vee\neg\psi) \preceq val(\neg(\varphi \wedge \psi))$. By ($neg_0^{\Rightarrow}$), $val(\neg\neg(\varphi \wedge \psi)) \preceq val(\neg(\neg\varphi \vee \neg\psi))$, and so, by ($neg_1^{\Rightarrow}$), $val(\varphi \wedge \psi) \preceq val(\neg(\neg\varphi \vee \neg\psi))$.
- (DM_2): By ($val_\vee$), $val(\varphi\vee\psi) = lub(\{val(\varphi), val(\psi)\})$, and so, by ($neg_0^{\Rightarrow}$), $val(\neg(\varphi \vee \psi)) \preceq val(\neg\varphi)$ and $val(\neg(\varphi \vee \psi)) \preceq val(\neg\psi)$. But, by ($val_\wedge$), $val(\neg\varphi\wedge\neg\psi) = glb(\{val(\neg\varphi), val(\neg\psi)\})$, and so $val(\neg(\varphi\vee\psi)) \preceq val(\neg\varphi \wedge \neg\psi)$.

■

Finally, for $\neg$ itself, we have from ($neg_1^{\Rightarrow}$):

THEOREM 2.18. *The rule of* double-negation introduction*:*

(DNI) $\varphi \models_{\mathbf{B}^0} \neg\neg\varphi$

is valid.

Proof. Immediate from ($\mathrm{neg}_1^{\Rightarrow}$).

■

Inspection of the proofs for (SIMP_0), (SIMP_1), (ADD_0), (ADD_1), (DM_0), (DM_1), (DM_2) and (DNI) shows that the result of substituting $\rightarrow$ for $\models_{\mathbf{B}^0}$ in them is a logical truth of $\mathbf{B}^0$.

We end here our brief survey of the properties of the basic tolerant logic $\mathbf{B}^0$. Even though, as has been seen, $\mathbf{B}^0$ already enshrines the core idea of tolerant logics, it is manifestly too weak a logic for a vague language, failing to satisfy many properties which any such logic may be reasonably expected to have. Hence, in the following we proceed to strengthen the logic in the usual fashion, by adding further and further constraints on its defining structures, while preserving its tolerance.

3. Classical Tolerant Logic

3.1. Towards the Classical Tolerant Logic $\mathbf{K}^0$

As far as vagueness is concerned, my favoured approach, for reasons concerning the philosophy of the logic of a vague language I cannot develop here, is, to speak somewhat loosely, to validate the full fragment of classical logic consistent with the naive theory of vagueness. To achieve that, we place further constraints on $\mathbf{B}^0$-structures, thereby characterizing the class of $\mathbf{K}^0$-*structures* (and, consequently, of $\mathbf{K}^0$-*models*), $\mathbf{K}^0$ being the "classical" tolerant logic generated by these structures. The investigation of the tolerant logics intermediate between $\mathbf{B}^0$ and $\mathbf{K}^0$ will have to wait for another occasion, but our stepwise way of proceeding will be sufficient to give a flavour of the variety of options available in this area.

We start by imposing the following constraints on D and T:

(D_1^2) If $v_0, v_1 \in D$, then $\mathrm{glb}(\{v_0, v_1\}) \in T$

(D is a *tolerance filter*—informally (and plausibly), if two vague pieces of information are very good, their conjunction is still at least good enough);

(D_2^2) For every $v_0, v_1 \in V$, if $\mathrm{lub}(\{v_0, v_1\}) \in D$, then either $v_0 \in T$ or $v_1 \in T$

(D is *tolerantly prime*—informally (and plausibly), the disjunction of two vague pieces of information can be very good only if at least one of them is at least good enough) and:

($D_3^{\Rightarrow}$) For every v, if $v \in D$, then $\mathrm{neg}(v) \in V \setminus T$;

($D_3^{\Leftarrow}$) For every v, if $\mathrm{neg}(v) \in V \setminus T$, then $v \in D$

(D is a *tolerance ultrafilter*—informally (and plausibly), a vague piece of information is very good iff its negation is not even good enough).[28]

We then require *distributivity of finite glbs over finite lubs*:[29]

(glb/lub$_1^2$) For every $v_0, v_1, v_2 \in V$, $\mathrm{glb}(\{v_0, \mathrm{lub}(\{v_1, v_2\})\}) = \mathrm{lub}(\{\mathrm{glb}(\{v_0, v_1\}), \mathrm{glb}(\{v_0, v_2\})\})$

—informally (and plausibly), the conjunction of a vague piece of information x with the disjunction of two vague pieces of information y and z is exactly as good as the disjunction of the conjunction of x with y and of the conjunction of x with z.

We finally add the following constraints on neg:

($\mathrm{neg}_0^{\Leftarrow}$) For every $v_0, v_1 \in V$, if $\mathrm{neg}(v_1) \preceq \mathrm{neg}(v_0)$, then $v_0 \preceq v_1$;

($\mathrm{neg}_1^{\Leftarrow}$) For every $v \in V$, $\mathrm{neg}(\mathrm{neg}(v)) \preceq v$

(of course, each will do given both ($\mathrm{neg}_0^{\Rightarrow}$) and ($\mathrm{neg}_1^{\Rightarrow}$))—informally (and plausibly), a vague piece of information is at least as good as the negation of its negation.[30]

---

[28]Note that, together with ($\mathrm{neg}_1^{\Rightarrow}$), ($D_3^{\Rightarrow}$) and ($D_3^{\Leftarrow}$) jointly entail that, for every $v \in V$, $v \in T \setminus D$ iff $\mathrm{neg}(v) \in T \setminus D$—informally (and plausibly), a vague piece of information is good enough but not very good iff its negation is good enough but not very good.

[29]Equivalent with *distributivity of finite lubs over finite glbs*:

(lub/glb$_1^2$) For every $v_0, v_1, v_2 \in V$, $\mathrm{lub}(\{v_0, \mathrm{glb}(\{v_1, v_2\})\}) = \mathrm{glb}(\{\mathrm{lub}(\{v_0, v_1\}), \mathrm{lub}(\{v_0, v_2\})\})$.

[30]Note that $\mathbf{K}^0$-structures as characterized need not be (and typically are not) Boolean algebras, differing from them in crucial respects. To give some examples, the system of designated and tolerated values of a $\mathbf{K}^0$-structure is collapsed in a Boolean algebra into the top value and negation in a $\mathbf{K}^0$-structure lacks the defining properties of a complementation operation in a Boolean algebra. More generally, nothing in our characterization of $\mathbf{K}^0$-structures forces their ordering relation to be such that it could be used to generate a Boolean algebra, provided suitable definitions of designatedness and of the negation operation (to give but the simplest example, a three-valued linear ordering with designated top value and tolerated middle value can be used to generate a $\mathbf{K}^0$-structure, but cannot be used to generate a Boolean algebra). In the following, in order to provide counterexamples I'll use a couple of structures whose ordering relation is such that it could in effect be used to generate a Boolean algebra, provided suitable definitions of designatedness and of the negation operation. Crucially, however, on such definitions the structures could not be used to provide the desired counterexamples (the reason being, roughly, that on such definitions it would be impossible to secure the designatedness of certain sentences). Hence, I will characterize these structures using alternative definitions of designatedness and of the negation operation.

DEFINITION 3.1. With all these further constraints in place on $\mathbf{K}^0$-structures, the consequence relation on pairs of sequences of wffs $\in WWF_{\mathscr{L}^0}$ constituting the classical tolerant logic $\mathbf{K}^0$ ($\models_{\mathbf{K}^0}$) can finally be defined as follows:

($\models_{\mathbf{K}^0}$) A sequence of wffs Δ is a consequence of a sequence of wffs Γ iff, for every $\mathbf{K}^0$-model $\mathfrak{M}$, if, for every $\varphi \in \mathrm{ran}(\Gamma)$, $\mathrm{val}_{\mathfrak{M}}(\varphi) \in D_{\mathfrak{M}}$, then, for some $\psi \in \mathrm{ran}(\Delta)$, $\mathrm{val}_{\mathfrak{M}}(\psi) \in T_{\mathfrak{M}}$.

3.2. The Strength of $\mathbf{K}^0$

We can then reap the harvest of the new semantics. We will do so by going through the newly introduced constraints focussing on their logical import.

From (D^2_1) and (D^2_2) we have:

THEOREM 3.2. *The rules of* adjunction *and* abjunction*:*

(ADJ) $\varphi, \psi \models_{\mathbf{K}^0} \varphi \wedge \psi$*;*

(ABJ) $\varphi \vee \psi \models_{\mathbf{K}^0} \varphi, \psi$

are valid.

PROOF.

- (ADJ): Immediate from (D^2_1).
- (ABJ): Immediate from (D^2_2).

■

(ADJ) and (ABJ) suffice to trigger new interesting failures of (T^l) and (T^r) for $\mathbf{K}^0$:

THEOREM 3.3. (ADJ) *and* (ABJ) *suffice to trigger* $\rightarrow$*-free failures of* (T^l) *and* (T^r) *for* $\mathbf{K}^0$*.*

PROOF.

- (T^l): Given (ADJ), $P_0, Q_0 \models_{\mathbf{K}^0} P_0 \wedge Q_0$ and $R_0, P_0 \wedge Q_0 \models_{\mathbf{K}^0} R_0 \wedge P_0 \wedge Q_0$. Setting $\Gamma = P_0, Q_0$, $\Delta = \varnothing$, $\Theta = P_0 \wedge Q_0$, $\Lambda = R_0$ and $\Xi = R_0 \wedge P_0 \wedge Q_0$, ($\mathrm{T}^l$) yields that $R_0, P_0, Q_0 \models_{\mathbf{K}^0} R_0 \wedge P_0 \wedge Q_0$, which is false, since there are $\mathbf{K}^0$-models where $\mathrm{val}(P_0)$, $\mathrm{val}(Q_0)$, $\mathrm{val}(R_0) \in D$, but $\mathrm{val}(P_0 \wedge Q_0) \in T \setminus D$ and $\mathrm{val}(R_0 \wedge P_0 \wedge Q_0) \in V \setminus T$.

- (T^r): Given (ABJ), $P_0 \vee Q_0 \vee R_0 \models_{\mathbf{K}^0} P_0 \vee Q_0, R_0$ and $P_0 \vee Q_0 \models_{\mathbf{K}^0} P_0, Q_0, R_0$. Moreover, $R_0 \models_{\mathbf{K}^0} P_0, Q_0, R_0$. Setting $\Xi = P_0 \vee Q_0 \vee R_0$, $\Lambda = \varnothing$, $\Theta = P_0 \vee Q_0, R_0$, $\Delta = \varnothing$ and $\Gamma = P_0, Q_0, R_0$, (T^r) yields that $P_0 \vee Q_0 \vee R_0 \models_{\mathbf{K}^0} P_0, Q_0, R_0$, which is false, since there are $\mathbf{K}^0$-models where $\mathrm{val}(P_0 \vee Q_0 \vee R_0) \in D$ and $\mathrm{val}(P_0 \vee Q_0) \in T \setminus D$, but $\mathrm{val}(P_0)$, $\mathrm{val}(Q_0)$, $\mathrm{val}(R_0) \in V \setminus T$.

■

By themselves, of course, (ADJ) and (ABJ) rule out, respectively, *subvaluational* and *supervaluational* approaches. In a transitive framework, these can be abstractly characterized as follows. A specification of a class of admissible models is assumed (usually, for applications to the problem of vagueness, these are just classical models). It is then said that a sequence of wffs Δ is a subvaluational (supervaluational) consequence of a sequence of wffs Γ iff, for every subclass S of admissible models, if, for every $\varphi \in \mathrm{ran}(\Gamma)$, for some (every) $\mathfrak{M} \in S$, φ has a designated value in $\mathfrak{M}$, then, for some $\psi \in \mathrm{ran}(\Delta)$, for some (every) $\mathfrak{M} \in S$, ψ has a designated value in $\mathfrak{M}$. Given this abstract characterization, the generalization to a non-transitive tolerant framework is straightforward. Given a tolerant consequence relation $\mathbf{T}^*$, we can define subvaluational and supervaluational versions of $\mathbf{T}^*$ by subvaluating and supervaluating on subclasses of $\mathbf{T}^*$-models as follows:

($\models^{sb}_{\mathbf{T}^*}$) A sequence of wffs Δ is a $\models^{sb}_{\mathbf{T}^*}$-consequence of a sequence of wffs Γ iff, for every subclass S of $\mathbf{T}^*$-models, if, for every $\varphi \in \mathrm{ran}(\Gamma)$, for some $\mathfrak{M} \in S$, $\mathrm{val}_{\mathfrak{M}}(\varphi) \in D_{\mathfrak{M}}$, then, for some $\psi \in \mathrm{ran}(\Delta)$, for some $\mathfrak{M} \in S$, $\mathrm{val}_{\mathfrak{M}}(\psi) \in T_{\mathfrak{M}}$;

($\models^{sp}_{\mathbf{T}^*}$) A sequence of wffs Δ is a $\models^{sp}_{\mathbf{T}^*}$-consequence of a sequence of wffs Γ iff, for every subclass S of $\mathbf{T}^*$-models, if, for every $\varphi \in \mathrm{ran}(\Gamma)$, for every $\mathfrak{M} \in S$, $\mathrm{val}_{\mathfrak{M}}(\varphi) \in D_{\mathfrak{M}}$, then, for some $\psi \in \mathrm{ran}(\Delta)$, for every $\mathfrak{M} \in S$, $\mathrm{val}_{\mathfrak{M}}(\psi) \in T_{\mathfrak{M}}$.

However, in this paper we will not investigate further subvaluational and supervaluational tolerant logics, returning instead to studying the properties of $\mathbf{K}^0$.

Defining a material-implication operator as usual:

DEFINITION 3.4. $\varphi \supset \psi := \neg\varphi \vee \psi$,[31]

from (D^2_2), $(D^{\Rightarrow}_3)$ and $(D^{\Leftarrow}_3)$ we have:

[31]The previous conventions about $\rightarrow$'s binding force apply to $\supset$ as well.

Theorem 3.5. *The rule of* modus ponens *and the* deduction theorem*:*

(MP$_\supset$) $\varphi \supset \psi, \varphi \models_{\mathbf{K}^0} \psi$;

(DT$_\supset$) *If* $\Gamma, \varphi \models_{\mathbf{K}^0} \psi$, *then* $\Gamma \models_{\mathbf{K}^0} \varphi \supset \psi$

are valid.

Proof.

- (MP$_\supset$): Suppose that $\mathrm{val}(\varphi \supset \psi), \mathrm{val}(\varphi) \in D$. Then, by ($D_2^2$), either $\mathrm{val}(\mathrm{neg}(\varphi)) \in T$ or $\mathrm{val}(\psi) \in T$. But $\mathrm{val}(\varphi) \in D$, and so, by ($D_3^{\Rightarrow}$), $\mathrm{val}(\mathrm{neg}(\varphi)) \notin T$. Therefore, $\mathrm{val}(\psi) \in T$.
- (DT$_\supset$): Suppose that $\Gamma, \varphi \models_{\mathbf{K}^0} \psi$. Either $\mathrm{val}(\varphi) \in D$ or not. In the first case, since $\Gamma, \varphi \models_{\mathbf{K}^0} \psi$, if, for every $\chi \in \mathrm{ran}(\Gamma)$, $\mathrm{val}(\chi) \in D$, $\mathrm{val}(\psi) \in T$, and so $\mathrm{lub}(\{\mathrm{val}(\mathrm{neg}(\varphi)), \mathrm{val}(\psi)\}) \in T$. In the second case, by ($\mathrm{neg}_1^{\Rightarrow}$), ($D_3^{\Rightarrow}$) and ($D_3^{\Leftarrow}$), $\mathrm{val}(\mathrm{neg}(\varphi)) \in T$, and so $\mathrm{lub}(\{\mathrm{val}(\mathrm{neg}(\varphi)), \mathrm{val}(\psi)\}) \in T$.

■

From ($D_3^{\Rightarrow}$) and ($D_3^{\Leftarrow}$) we also have:

Theorem 3.6. *The properties of* negation in the premises *and* negation in the conclusions*:*

(NP) *If* $\Gamma \models_{\mathbf{K}^0} \Delta, \varphi$, *then* $\Gamma, \neg\varphi \models_{\mathbf{K}^0} \Delta$;

(NC) *If* $\Gamma, \varphi \models_{\mathbf{K}^0} \Delta$, *then* $\Gamma \models_{\mathbf{K}^0} \Delta, \neg\varphi$

hold.

Proof.

- (NP): Suppose that $\Gamma \models_{\mathbf{K}^0} \Delta, \varphi$. Either $\mathrm{val}(\varphi) \in T$ or not. In the first case, by ($\mathrm{neg}_1^{\Rightarrow}$) and ($D_3^{\Rightarrow}$), $\mathrm{val}(\mathrm{neg}(\varphi)) \notin D$, and so it is vacuously true that, if, for every $\psi \in \mathrm{ran}(\Gamma, \neg\varphi)$, $\mathrm{val}(\psi) \in D$, then, for some $\psi \in \mathrm{ran}(\Delta)$, $\mathrm{val}(\psi) \in T$. In the second case, since $\Gamma \models_{\mathbf{K}^0} \Delta, \varphi$, it is true that, if, for every $\psi \in \mathrm{ran}(\Gamma, \neg\varphi)$, $\mathrm{val}(\psi) \in D$, then, for some $\psi \in \mathrm{ran}(\Delta)$, $\mathrm{val}(\psi) \in T$.
- (NC): Suppose that $\Gamma, \varphi \models_{\mathbf{K}^0} \Delta$. Either $\mathrm{val}(\varphi) \in D$ or not. In the first case, since $\Gamma, \varphi \models_{\mathbf{K}^0} \Delta$, it is true that, if, for every $\psi \in \mathrm{ran}(\Gamma)$, $\mathrm{val}(\psi) \in D$, then, for some $\psi \in \mathrm{ran}(\Delta, \neg\varphi)$, $\mathrm{val}(\psi) \in T$. In the second case, by ($D_3^{\Leftarrow}$), $\mathrm{val}(\mathrm{neg}(\varphi)) \in T$, and so it is true that, if, for every $\psi \in \mathrm{ran}(\Gamma)$, $\mathrm{val}(\psi) \in D$, then, for some $\psi \in \mathrm{ran}(\Delta, \neg\varphi)$, $\mathrm{val}(\psi) \in T$.

∎

THEOREM 3.7. *The law of* excluded middle *and the attendant property of* exhaustion *for the consequence relation:*

(LEM) $\varnothing \models_{\mathbf{K}^0} \varphi \vee \neg\varphi$;

(EXH) $\varnothing \models_{\mathbf{K}^0} \varphi, \neg\varphi$

are valid.

PROOF.

- (LEM): Either $\text{val}(\varphi) \in D$ or not. In the first case, $\text{lub}(\{\text{val}(\varphi), \text{val}(\neg\varphi)\}) \in D$, and so *a fortiori* $\text{lub}(\{\text{val}(\varphi), \text{val}(\neg\varphi)\}) \in T$. In the second case, by ($\text{neg}_1^{\Rightarrow}$), ($D_3^{\Rightarrow}$) and ($D_3^{\Leftarrow}$), $\text{val}(\neg\varphi) \in T$, and so $\text{lub}(\{\text{val}(\varphi), \text{val}(\neg\varphi)\}) \in T$.
- (EXH): Immediate from the proof of (LEM).

∎

THEOREM 3.8. *The law of* non-contradiction *and the attendant property of* explosion *for the consequence relation:*

(LNC) $\varphi \wedge \neg\varphi \models_{\mathbf{K}^0} \varnothing$;

(EXP) *If* $\varphi, \neg\varphi \models_{\mathbf{K}^0} \varnothing$

are valid.

PROOF.

- (LNC): Suppose for *reductio* that $\text{val}(\varphi \wedge \neg\varphi) \in D$. Then $\text{val}(\varphi) \in D$ and $\text{val}(\neg\varphi) \in D$. But, if $\text{val}(\varphi) \in D$, by ($D_3^{\Rightarrow}$), $\text{val}(\neg\varphi) \notin D$.
- (EXP): Immediate from the proof of (LNC).[32]

∎

From ($\text{glb}/\text{lub}_1^2$) we have:

THEOREM 3.9. *The distributivity rules:*

[32] Note that (LEM) and (EXH) ((LNC) and (EXP)) would come apart in the supervaluational (subvaluational) consequence relation $\models^{sp}_{\mathbf{K}^0}$ ($\models^{sb}_{\mathbf{K}^0}$) which uses $\mathbf{K}^0$-models as the class of admissible models.

(DISTR$_{\wedge/\vee}$) $\varphi \wedge (\psi \vee \chi) \models_{\mathbf{K}^0} (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$;
(DISTR$_{\vee/\wedge}$) $\varphi \vee (\psi \wedge \chi) \models_{\mathbf{K}^0} (\varphi \vee \psi) \wedge (\varphi \vee \chi)$

are valid.

Proof.

- (DISTR$_{\wedge/\vee}$): Immediate from (glb/lub$^2_1$).
- (DISTR$_{\vee/\wedge}$): Immediate from (lub/glb$^2_1$).

■

From either (neg$_0^{\Leftarrow}$) or (neg$_1^{\Leftarrow}$) we have:

Theorem 3.10. *The rule of* double-negation elimination*:*

(DNE) $\neg\neg\varphi \models_{\mathbf{K}^0} \varphi$

is valid.

Proof. Immediate from either (neg$_0^{\Leftarrow}$) or (neg$_1^{\Leftarrow}$).

■

Theorem 3.11. *The* De Morgan *rule:*

(DM$_3$) $\neg(\varphi \wedge \psi) \models_{\mathbf{K}^0} \neg\varphi \vee \neg\psi$

is valid.

Proof. By (val$_\vee$), $\mathrm{val}(\neg\varphi \vee \neg\psi) = \mathrm{lub}(\{\mathrm{val}(\neg\varphi), \mathrm{val}(\neg\psi)\})$, and so, by (neg$_0^{\Rightarrow}$), $\mathrm{val}(\neg(\neg\varphi \vee \neg\psi)) \preceq \mathrm{val}(\neg\neg\varphi)$ and $\mathrm{val}(\neg(\neg\varphi \vee \neg\psi)) \preceq \mathrm{val}(\neg\neg\psi)$. By either (neg$_0^{\Leftarrow}$) or (neg$_1^{\Leftarrow}$), $\mathrm{val}(\neg(\neg\varphi \vee \neg\psi)) \preceq \mathrm{val}(\varphi)$ and $\mathrm{val}(\neg(\neg\varphi \vee \neg\psi)) \preceq \mathrm{val}(\psi)$. But, by (val$_\wedge$), $\mathrm{val}(\varphi \wedge \psi) = \mathrm{glb}(\{\mathrm{val}(\varphi), \mathrm{val}(\psi)\})$, and so $\mathrm{val}(\neg(\neg\varphi \vee \neg\psi)) \preceq \mathrm{val}(\varphi \wedge \psi)$. By (neg$_0^{\Rightarrow}$), $\mathrm{val}(\neg(\varphi \wedge \psi)) \preceq \mathrm{val}(\neg\neg(\neg\varphi \vee \neg\psi))$, and so, by either (neg$_0^{\Leftarrow}$) or (neg$_1^{\Leftarrow}$), $\mathrm{val}(\neg(\varphi \wedge \psi)) \preceq \mathrm{val}(\neg\varphi \vee \neg\psi)$.

■

Possibly with exception of (NC), (LEM) and (EXH) (whose defence I cannot undertake here), I submit that all the various rules and laws we have been reviewing can very plausibly be taken to represent correct patterns of reasoning with a vague language: competent speakers do usually abide by—and hold others responsible to—them. It is thus crucial to see how, in $\mathbf{K}^0$, the desired restrictions on (T^l) and (T^r) can be achieved while preserving the full validity of these other rules and laws.

3.3. The Weakness of $\mathbf{K}^0$

I hope that the foregoing is sufficient to show how rich a fragment of classical logic is preserved by $\mathbf{K}^0$. What is not preserved are exactly those rules and laws of the operators which encode the transitivity of the classical consequence relation (that some such rules and laws exist is evident from the eliminability of the cut rule in a standard sequent calculus for classical logic). These can be made to emerge already in the extensional fragment $\mathbf{K}^0_{\not\rightarrow}$ of $\mathbf{K}^0$ (that is, the restriction of $\mathbf{K}^0$ to the extensional language $\mathscr{L}^0_{\not\rightarrow}$ such that $WFF_{\mathscr{L}^0_{\not\rightarrow}} = WFF_{\mathscr{L}^0} \setminus \{\varphi :\rightarrow \text{ occurs in } \varphi\}$). We have already seen (theorem 3.3) that $\mathbf{K}^0_{\not\rightarrow}$ fails to satisfy (T^l). Correspondingly:

THEOREM 3.12. *The property of* material implication in the premises*:*

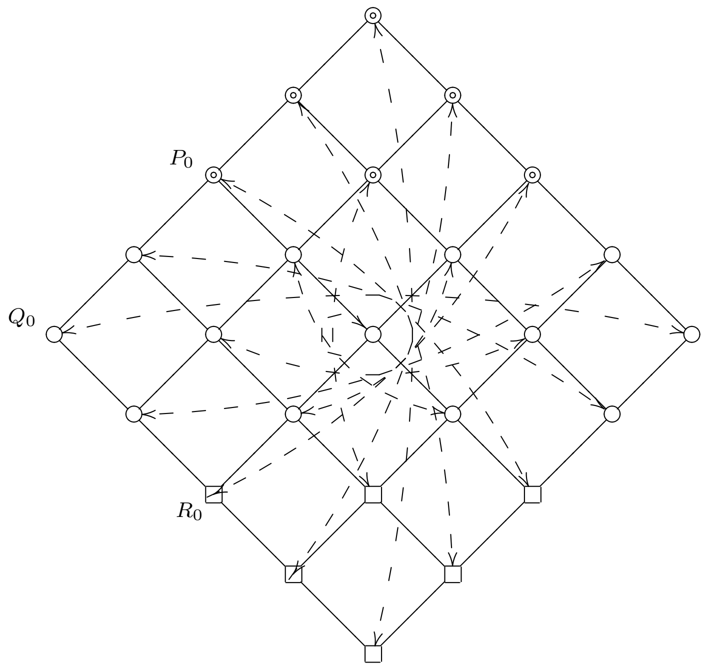
$(\mathrm{IP}_\supset)$ *If* $\Gamma \models_{\mathbf{K}^0_{\not\rightarrow}} \Delta, \varphi$ *and* $\Theta, \psi \models_{\mathbf{K}^0_{\not\rightarrow}} \Lambda$*, then* $\Theta, \Gamma, \varphi \supset \psi \models_{\mathbf{K}^0_{\not\rightarrow}} \Lambda, \Delta$

fails.

PROOF. We consider the $\mathbf{K}^0_{\not\rightarrow}$-model $\mathfrak{M}_0$, where:

- $V_{\mathfrak{M}_0} = \{\langle x, y\rangle : 0 \leq x \leq 4,\ 0 \leq y \leq 4\}$;
- $D_{\mathfrak{M}_0} = \{\langle x, y\rangle : x + y \geq 6\}$;
- $\preceq_{\mathfrak{M}_0} = \{\langle \pi_0, \pi_1\rangle : 0\mathrm{co}(\pi_0) \leq 0\mathrm{co}(\pi_1) \text{ and } 1\mathrm{co}(\pi_0) \leq 1\mathrm{co}(\pi_1)\}$ (where $i\mathrm{co}(\pi)$ is the ith coordinate of the ordered pair π);
- $\mathrm{tol}_{\mathfrak{M}_0} = \{\langle \pi_0, \Pi\rangle : \Pi = \{\pi_1 : [0\mathrm{co}(\pi_1) \geq 0\mathrm{co}(\pi_0) - 3 \text{ and } 1\mathrm{co}(\pi_1) \geq 1\mathrm{co}(\pi_0)] \text{ or } [0\mathrm{co}(\pi_1) \geq 0\mathrm{co}(\pi_0) \text{ and } 1\mathrm{co}(\pi_1) \geq 1\mathrm{co}(\pi_0) - 3]\}\}$;
- $O_{\mathfrak{M}_0} = \{\mathrm{neg}_{\mathfrak{M}_0}\}$ (where $\mathrm{neg}_{\mathfrak{M}_0} = \{\langle \pi_0, \pi_1\rangle : [i\mathrm{co}(\pi_1) = 0 \text{ iff } i\mathrm{co}(\pi_0) = 4] \text{ and } [i\mathrm{co}(\pi_1) = 1 \text{ iff } i\mathrm{co}(\pi_0) = 3] \text{ and } [i\mathrm{co}(\pi_1) = 2 \text{ iff } i\mathrm{co}(\pi_0) = 2] \text{ and } [i\mathrm{co}(\pi_1) = 3 \text{ iff } i\mathrm{co}(\pi_0) = 1]\}$);
- $\mathrm{int}_{\mathfrak{M}_0}(P_0) = \langle 2, 4\rangle$, $\mathrm{int}_{\mathfrak{M}_0}(Q_0) = \langle 0, 4\rangle$ and $\mathrm{int}_{\mathfrak{M}_0}(R_0) = \langle 0, 2\rangle$.

Setting $\Gamma = P_0$, $\Delta = \varnothing$, $\varphi = P_0$, $\Theta = Q_0 \supset R_0$, $\psi = Q_0$ and $\Lambda = R_0$, it's easy to check that the consequent of $(\mathrm{IP}_\supset)$ is falsified by $\mathfrak{M}_0$ even though its antecedent holds. $\mathfrak{M}_0$ may be depicted by the following Hasse diagram (where double circular nodes indicate the members of $D_{\mathfrak{M}_0}$, simple circular nodes the members of $T_{\mathfrak{M}_0}$, square nodes the members of $V_{\mathfrak{M}_0} \setminus T_{\mathfrak{M}_0}$ and dashed arrows indicate the negation operation):



■

We have also seen (theorem 3.3) that $\mathbf{K}^0_{\not\twoheadrightarrow}$ fails to satisfy (T^r). Correspondingly:

THEOREM 3.13. *The property of* disjunction in the premises*:*

(DP) *If* $\Gamma, \varphi \models_{\mathbf{K}^0_{\not\twoheadrightarrow}} \Delta$ *and* $\Theta, \psi \models_{\mathbf{K}^0_{\not\twoheadrightarrow}} \Lambda$*, then* $\Theta, \Gamma, \varphi \vee \psi \models_{\mathbf{K}^0_{\not\twoheadrightarrow}} \Lambda, \Delta$

fails.

PROOF. With $\mathfrak{M}_0$ as in the proof of theorem 3.12, setting $\Gamma = P_0$, $\varphi = \neg P_0$, $\Delta = R_0$, $\Theta = Q_0 \supset R_0$, $\psi = Q_0$ and $\Lambda = R_0$, it's easy to check that the consequent of (DP) fails in $\mathfrak{M}_0$ even though its antecedent holds.

■

4. The Consistency of the Naive Theory of Vagueness

4.1. Consistency in a Tolerant Framework

Building on what seems to be a plausible model of our use of a vague language, we constructed the basic tolerant logic $\mathbf{B}^0$. We then proceeded to develop stronger and stronger systems in order to capture more and more of what seem to be correct patterns of reasoning with a vague language, while at the same time taking care of preserving the hallmark of the weakness of a tolerant logic—namely, the non-transitivity of the consequence relation. The result has been the logic $\mathbf{K}^0$. It is now time to show that $\mathbf{K}^0$ is in effect suitable to fulfil the theoretical task for which it has been developed: that of making the naive theory of vagueness consistent once it is assumed as the background logic of the theory.

The informal presentation of the naive theory of vagueness left it rather unspecific what it exactly amounts to. Choosing a particular example, we can be more precise. Keeping in mind the expressive limitations of $\mathscr{L}^0$, I propose to consider as a simple paradigmatic example of a zeroth-order naive theory of vagueness the theory $\mathcal{N}^0$ based on the following axioms (where P_i translates into $\mathscr{L}^0$ the English 'i is a small natural number' and this is used in a context where 0 is an indisputable positive case and 2 an indisputable negative case):

(N^{0^p}) P_0;

(N^{0^n}) $\neg P_2$;

($\mathrm{N}^{0^{t_0}}$) $P_0 \supset P_1$;

($\mathrm{N}^{0^{t_1}}$) $P_1 \supset P_2$.

$\mathcal{N}^0$ is palpably a zeroth-order naive theory of vagueness inasmuch as it contains all the three characteristic claims of such a theory (see section 1): the existence of positive cases, the existence of negative cases and the non-existence of a sharp boundary between them.

Given the non-standard properties of $\models_{\mathbf{K}^0}$ and the existence of two non-coincident kinds of "good" values in the underlying semantics (the members of D and the members of T), it is not immediate how the intuitive notion of *consistency* might best be captured in the present framework. Indeed, it is easily seen that the framework allows for a multiplicity of different, non-equivalent definitions of the consistency of a sequence. Such definitions will differ in their strength and there is every reason to think that different definitions will prove useful for different theoretical purposes. In this paper, I propose however to focus for simplicity's sake only on one such definition which appears to be very natural and deeply embedded in our inferential practices. Consider that, given a consequence relation $\mathbf{L}$ exhibiting all the usual structural properties possibly with the exception of (T^l) and (T^r), the following principle linking consequence with *acceptance* and *rejection* should hold:

(AR) $\Gamma \vdash_{\mathbf{L}} \Delta$ iff, on logical grounds, one ought not to accept all the coordinates of Γ and reject all the coordinates of Δ.

If Δ is empty, (AR) comes down to the condition that, on logical grounds, one ought not to accept all the coordinates of Γ, which in turn seems to capture well at least one construal of the intuitive notion of inconsistency. On such a reading, a sequence Γ is inconsistent in a logic $\mathbf{L}$ iff $\Gamma \vdash_{\mathbf{L}} \varnothing$ (and so Γ is consistent in $\mathbf{L}$ iff $\Gamma \nvdash_{\mathbf{L}} \varnothing$). Notice that a parallel argument can be run for the notion of *validity*: if Γ is empty, (AR) comes down to the condition that, on logical grounds, one ought not to reject all the coordinates of Δ, which in turn seems to capture well at least one construal of the intuitive notion of validity. On such a reading, a sequence Δ is valid in a logic $\mathbf{L}$ iff $\varnothing \vdash_{\mathbf{L}} \Delta$.

Applying these definitions to $\mathbf{K}^0$ (and, more generally, to any logic definable in our framework), we obtain that ($\models_{\mathbf{K}^0}$) reduces the consistency of Γ to every coordinate of Γ being in $D_{\mathfrak{M}}$ for some $\mathbf{K}^0$-model $\mathfrak{M}$, and that it reduces the validity of Δ to some coordinate of Δ being in $T_{\mathfrak{M}}$ for every $\mathbf{K}^0$-model $\mathfrak{M}$. Given the properties of $\models_{\mathbf{K}^0}$, the *theory* based on Γ (that is, the set of (single-conclusion) logical consequences of Γ)[33] will then be guaranteed to have all of its members in $T_{\mathfrak{M}}$.

Note in particular that, given ($\models_{\mathbf{K}^0}$), premises (and conclusions) are not really "put together" when evaluating whether consequence holds or not: what is relevant in such evaluation is only whether, for every premise, the value of the premise belongs to D rather than whether the value resulting from conjunctively "putting together" the premises (the glb of the set of values of the premises) itself belongs to D (analogously, what is relevant in such evaluation is only whether, for some conclusion, the value of the conclusion belongs to T rather than whether the value resulting from disjunctively "putting together" the conclusions (the lub of the set of values of the conclusions) itself belongs to T). This independence seems desirable, as the requirement that the values of the premises (or of the conclusions)

[33] A more general notion of theory exploiting the multiple-conclusion setting would have the theory of Γ be the set of sequences which are consequences of Γ. I skip over such niceties here. Notice also that, in a non-transitive framework, we cannot sensibly employ the usual, stronger definition of the theory of Γ, which identifies it with the *closure* of Γ under logical consequence (that is, the smallest set of sentences $\mathcal{T}$ such that:

(i) For every φ, if $\Gamma \vdash \varphi$, then $\varphi \in \mathcal{T}$;

(ii) For every φ and Θ (such that $\mathrm{ran}(\Theta) = \mathcal{T}$), if $\Theta \vdash \varphi$, then $\varphi \in \mathcal{T}$).

I discuss the philosophical and technical implications of this circumstance in [16].

undergo any logical operation (such as glb or lub) before being evaluated for consequence would seem to build already into the very definition of the consequence relation a form of transitivity contrary to the spirit of tolerant logics. Therefore, as regards consistency in particular, ($\models_{\mathbf{K}^0}$) only requires there to be a $\mathbf{K}^0$-model $\mathfrak{M}$ where the value of every premise belongs to $D_{\mathfrak{M}}$—it does not require there to be a $\mathbf{K}^0$-model $\mathfrak{M}$ where the value of the conjunction of every premise itself belongs to $D_{\mathfrak{M}}$ (which, given the characteristic "lowering" behaviour that the conjunction operation has in $\mathbf{K}^0$, would amount to a more exacting requirement).[34] It thus only remains to show that there is indeed such a model.

4.2. A Model of Tolerance

For simplicity's sake, we focus on the consistency result for $\mathbf{K}^0_{\not\leftrightarrow}$. The extension to full $\mathbf{K}^0$ is straightforward.

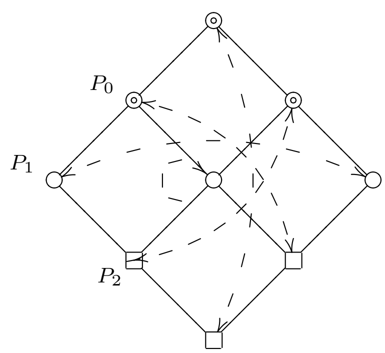
THEOREM 4.1. *The axiomatic base of* $\mathcal{N}^0$ $\langle(\mathrm{N}^{0^p})$, (N^{0^n}), $(\mathrm{N}^{0^{t_0}})$, $(\mathrm{N}^{0^{t_1}})\rangle$ *is consistent in* $\mathbf{K}^{\mathbf{0}}_{\not\leftrightarrow}$.

PROOF. We consider the $\mathbf{K}^0_{\not\leftrightarrow}$-model $\mathfrak{C}^0$ where:

- $V_{\mathfrak{C}^0} = \{\langle x, y\rangle : 0 \leq x \leq 2,\ 0 \leq y \leq 2\}$;
- $D_{\mathfrak{C}^0} = \{\langle x, y\rangle : x + y \geq 3\}$;
- $\preceq_{\mathfrak{C}^0} = \{\langle \pi_0, \pi_1\rangle : 0\mathrm{co}(\pi_0) \leq 0\mathrm{co}(\pi_1) \text{ and } 1\mathrm{co}(\pi_0) \leq 1\mathrm{co}(\pi_1)\}$;
- $\mathrm{tol}_{\mathfrak{C}^0} = \{\langle \pi_0, \Pi\rangle : \Pi = \{\pi_1 : [0\mathrm{co}(\pi_1) \geq 0\mathrm{co}(\pi_0) - 1 \text{ and } 1\mathrm{co}(\pi_1) \geq 1\mathrm{co}(\pi_0)] \text{ or } [0\mathrm{co}(\pi_1) \geq 0\mathrm{co}(\pi_0) \text{ and } 1\mathrm{co}(\pi_1) \geq 1\mathrm{co}(\pi_0) - 1]\}\}$;
- $O_{\mathfrak{C}^0} = \{\mathrm{neg}_{\mathfrak{C}^0}\}$, where $\mathrm{neg}_{\mathfrak{C}^0} = \{\langle \pi_0, \pi_1\rangle : [i\mathrm{co}(\pi_1) = 0 \text{ iff } i\mathrm{co}(\pi_0) = 2] \text{ and } [i\mathrm{co}(\pi_1) = 1 \text{ iff } i\mathrm{co}(\pi_0) = 1]\}$;
- $\mathrm{int}_{\mathfrak{C}^0}(P_0) = \langle 1, 2\rangle$, $\mathrm{int}_{\mathfrak{C}^0}(P_1) = \langle 0, 2\rangle$ and $\mathrm{int}_{\mathfrak{C}^0}(P_2) = \langle 0, 1\rangle$.

It's easy to check that $\mathfrak{C}^0$ is indeed a $\mathbf{K}^0_{\not\leftrightarrow}$-model for the axiomatic base of $\mathcal{N}^0$. $\mathfrak{C}^0$ may be depicted by the following Hasse diagram (notational conventions as in proof 3.12):

[34] This is not to say of course that such consequence relations lack theoretical interest. Their study must however wait for another occasion.



■

5. Conclusion and Glimpses Beyond

We have thus seen how a fairly natural weakening of the logic, targeting one of the structural properties of the consequence relation rather than any property pertaining to a specific logical constant, is sufficient to stabilize the sentential fragment of the naive theory of vagueness. I think that this result is crucial in reinstating the naive theory as one of the main competitors in the vagueness debate, especially in the presence of the significant shortcomings of the other extant theories.

Some of the many *philosophical* issues arising from the consistency result have been briefly mentioned in the text—their treatment, necessary as it is for a full vindication of the naive theory, must wait for another occasion (see [16]). More *technical* issues will also have to be investigated elsewhere. Chief among them is the extension of tolerant logics (and $\mathbf{K}^0$ in particular) to the first-order level and the theory of identity, extension which can be anticipated to be non-trivial and yet crucial for a satisfactorily comprehensive theory of vagueness (see [18]). Other tasks will consist in providing tolerant logics (and $\mathbf{K}^0$ in particular) with adequate deductive systems and with an alternative possible-world semantics. A whole set of philosophical and technical issues is thus generated by the non-transitivist solution to the sorites paradox presented in this paper. I hope here to have motivated and articulated the solution well enough as to make this research program appear worth pursuing further.

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